Modeling Groundwater Flow

An Honors Thesis (HONRS 499)

by

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Abstract

The purpose of this thesis is to use basic ideas of Mathematics and Physics, such as Darcy's Law and the Law of Conservation of Mass to derive a differential equation to model the flow of groundwater. This equation will then be solved analytically for the two-dimensional groundwater flow equation, which will be a function that describes the head height of groundwater at a certain point at any given time. The differential equation will also be solved by a numerical method, using basic laws of Calculus. The two methods – analytical and numerical – will then be compared by applying each one to an example problem, and another possible method for solving the example problem will arise from the comparison.

Acknowledgements

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Also, to the Botswin "Lounge Rats," who probably never want to hear the word 'thesis' again, thanks for all the support you gave me.

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1. Introduction

If you dig into the ground anywhere on earth, you will eventually hit water. It may not appear to be moving, but it is, likely very slowly. The study and modeling of groundwater is necessary because we rely on groundwater for many things, such as irrigating crops and supplying us with drinking-water, and it is important to know how to predict where that water will be found, how deep we need to dig for it, and how long it might stay there. For example, if a person wants to build a house somewhere outside of a city, they need to consider where they will be able to put a well in to get water for their house, how deep that well will need to be, and how likely it is that the well will produce a good water supply for a long period of time. Drilling wells is an expensive and time consuming process, so we would not want to just drill haphazardly hoping to find water somewhere, we want to know ahead of time where drilling will be the most productive.

In 1850 a French engineer, Henri Darcy, was interested in studying how water flows through the ground, because he wanted to set up a water filtering system for the city of Dijon, France, that would use beds of clean sand to filter the water. By experimentation, Darcy developed an equation to model how groundwater flows, called *Darcy's Law.* Darcy's Law gives a relationship between the *volumetric flow rate* of water, Q, and the *hydraulic head,* h, which can be thought of as the height of the water level, in units of length, measured relative to some chosen level, such as sea level, in terms of the change in head height between two particular wells, Δh, the distance the water was traveling, L, and the cross-sectional area the water was flowing through, A. Darcy's law says that the volumetric flow rate is equal to the *flux* (the volumetric flow rate through a cross-sectional area of one unit in units of length/time), denoted q, times a cross-sectional area. The flux, q is also equal to the *hydraulic conductivity,* K(units of length/time, which gives a measure of how well groundwater flows through a given material (sand, for example, through which water flows quite easily, has a range of K values from 10^{-2} to 10^{3} ft/day, whereas clay, through which water does not flow easily, has a range of K values from 10^{-7} to 10^{-3} ft/day (Hadlock, 26)), times the gradient of the hydraulic head, ∇h, (Hadlock, 36-37).

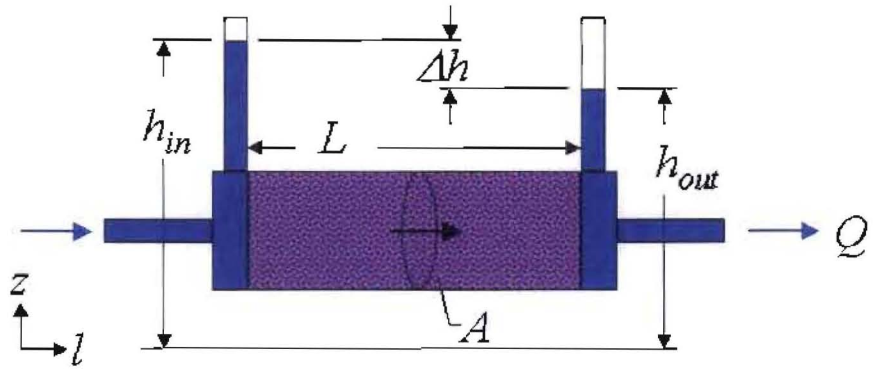

Figure 1: A simple column to model the one-dimensional flow of water through an aquifer. Water flows in the left end and out the right end of a cylinder filled with geologic material such as sand. Hydraulic head levels are measured at wells located at each end of the column. (Brown)

For one-dimensional flow, as in Figure 1, in the x-direction, Darcy's Law can be written, in the discrete case, as

$$Q = KiA \qquad \text{where } i = \frac{\Delta h}{L}, \tag{1}$$

or, for the continuous case when we are looking at a single position along the aquifer, as

$$Q = KiA \qquad \text{where } i = -\frac{dh}{dx}. \tag{2}$$

For this thesis, we will assume that we are dealing with *isotropic* geologic material, such as sand, in which, at any particular location, the hydrologic properties in any direction are identical. We also assume that the *aquifer* (a portion of the ground through which water can move relatively easily) is relatively flat, so that the only significant direction of groundwater flow is horizontally. (Hadlock, 183) We choose not to look at the case of *anisotropic* materials, such as granite, which are more complex to model, because the hydrologic properties vary in any direction.

The *Law of Conservation of Mass* (Wang and Anderson, 67) says that mass cannot be created or destroyed, and therefore the amount of mass in any closed system cannot change regardless of what is happening within that system. Using Darcy's Law and the Law of Conservation of Mass, a differential equation, which we call the *groundwater flow equation*, will be derived and solved. This equation describes groundwater flow and can be used to calculate the hydraulic head at any point in an aquifer, at any given time. A numerical method for solving the steady-state problem, which describes the long time behavior of groundwater levels is also discussed and compared to the analytical method for solving the same problem and finally, a combination of the two methods is introduced as a possible solution method.

2. Derivation of two-dimensional groundwater flow equation

To derive the two-dimensional groundwater flow equation, we apply the Law of Conservation of Mass to a small rectangular solid within a larger region below the Earth's surface that contains groundwater, of width $2\Delta y$, length $2\Delta x$, and height $\Delta z = 1$, with sides parallel to the coordinate planes (xy, xz, and yz). We assume that the flow in the z-direction is negligible, so we consider only the flow in the x- and y-directions.

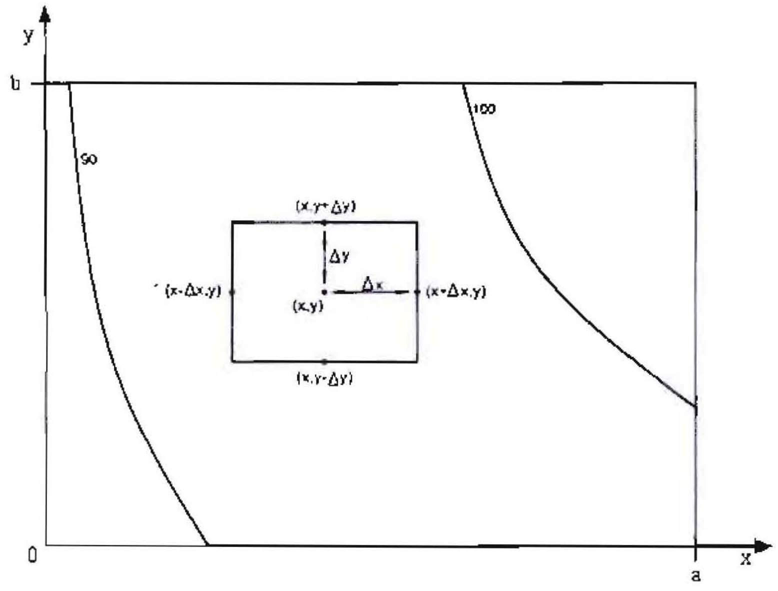

Figure 2: Volume increment in a ground-water flow field with two-dimensional flow. (Hadlock, 197)

The Law of Conservation of Mass, for the case of a fluid, says that

> Flow in left face + flow in right face + flow in bottom face + flow in top face + rate of water added or removed from storage = 0, (3)

(Wang and Anderson, 67) where we assume that along the left face, for example, the inward flow of water, which may vary as the y-coordinate varies, can be represented by the one specific flow of water at the center of the left face, namely $Q(x-\Delta x, y)$ and make similar assumptions for the other three faces of the rectangle (Hadlock, 197). Then Eq. (3) becomes

$$\begin{aligned} &Q(x-\Delta x,y)-Q(x+\Delta x,y)+Q(x,y-\Delta y)-Q(x,y+\Delta y) \\ &\quad = R(x,y)*(4\Delta x\Delta y)-S*(4\Delta x\Delta y)*\frac{\Delta h}{\Delta t}, \end{aligned} \tag{4}$$

where $Q(x,y)$ is the volumetric flow rate (units of volume/time), $R(x,y)$ is the volume of water added per unit time per unit aquifer area and S is the volume of water released from storage per unit area of aquifer per unit decline in head, or $S=-\frac{\Delta V_w}{\Delta x\Delta y\Delta h}$, where $\frac{\Delta V_w}{\Delta t}$ is the rate of release of a volume of water, ΔV_w, from storage (Wang and Anderson, 68).

Using the two dimensional version of Darcy's Law (Hadlock, 190),

$$Q = qA = -K\nabla h A,$$

which is equivalent to saying

$$Q_{x\text{-direction}} = -K\frac{\partial h}{\partial x}A,$$

and

$$Q_{y\text{-direction}} = -K\frac{\partial h}{\partial y}A,$$

which are simply the one-dimensional versions of Darcy's Law for each direction, we can find a relation between the volumetric flow rate Q and the hydraulic head, which is what we really want. Applying Darcy's Law, Eq. (4) becomes

$$-K\left[\frac{\partial h}{\partial x}(x-\Delta x, y)(2\Delta y) - \frac{\partial h}{\partial x}(x+\Delta x, y)(2\Delta y) + \frac{\partial h}{\partial y}(x, y-\Delta y)(2\Delta x) - \frac{\partial h}{\partial y}(x, y+\Delta y)(2\Delta x)\right]$$

$$= R(x,y)*(4\Delta x\Delta y) - S*(4\Delta x\Delta y)*\frac{\Delta h}{\Delta t}$$

Dividing through by $-4K\Delta x\Delta y$ gives

$$\frac{\frac{\partial h}{\partial x}(x+\Delta x, y) - \frac{\partial h}{\partial x}(x-\Delta x, y)}{2\Delta x} + \frac{\frac{\partial h}{\partial y}(x, y+\Delta y) - \frac{\partial h}{\partial y}(x, y-\Delta y)}{2\Delta y} = \frac{1}{K}\left(S\frac{\Delta h}{\Delta t} - R(x,y,t)\right).$$

Finally, taking the limit as $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta t \to 0$ we get the groundwater flow equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{1}{K}\left(S\frac{\partial h}{\partial t} - R(x,y,t)\right), \tag{5}$$

which is a partial differential equation in which the unknown function to be solved for is the hydraulic head $h(x,y,t)$.

3. Solving the two-dimensional flow equation

We begin with the partial differential equation (5) for the hydraulic head which was derived in the previous section,

$$h_{xx} + h_{yy} = \frac{1}{k}h_t, \qquad 0 < x < a, \;\; 0 < y < b, \;\; t > 0, \tag{6}$$

where $k = \frac{K}{S}$ is the *hydraulic diffusivity* – the ratio of hydraulic conductivity to storage coefficient - and it is assumed that there is no water being added by letting $R(x,y,t) = 0$ for all x, y, and t.

In order to get a unique solution to Eq. (6), we need to impose additional conditions on the function $h(x,y,t)$. In this case, the appropriate choices are boundary conditions and an initial condition. If we know the groundwater levels at the boundaries of the region in Figure 2, $x=0$, $x=a$, $y=0$, and $y=b$, we can write the boundary conditions as functions $f_1(x)$, $f_2(x)$, $g_1(y)$, and $g_2(y)$, respectively,

$$h(x,0,t) = f_1(x) \qquad 0 < x < a, \quad t > 0, \tag{7}$$

$$h(x,b,t) = f_2(x) \qquad 0 < x < a, \quad t > 0, \tag{8}$$

$$h(0,y,t) = g_1(y) \qquad 0 < y < b, \quad t > 0, \tag{9}$$

$$h(a,y,t) = g_2(y) \qquad 0 < y < b, \quad t > 0. \tag{10}$$

If we know the initial groundwater level at each point (x,y) at time $t=0$, we can write the initial condition in terms of the function $f(x,y)$, which describes the initial heights, i.e.

$$h(x,y,0) = f(x,y), \qquad 0 < x < a,\ 0 < y < b. \tag{11}$$

Problem (6)-(11) is known as an *initial value-boundary value problem* (Powers, 118).

In order to make this problem simpler to solve, we split it into two separate problems, the *steady state problem* and the *transient problem* (Powers, 123). As $t \to \infty$, we expect that $h(x,y,t)$ will no longer vary with time, so

$$\lim_{t\to\infty} h(x,y,t) = v(x,y),$$

and

$$\lim_{t\to\infty} h_t(x,y,t) = 0,$$

where $v(x,y)$ is called the *steady-state solution*. Using these limits in Eqs. (6)-(10), we can see that $v(x,y)$ solves the boundary value problem

$$v_{xx} + v_{yy} = 0, \qquad 0 < x < a,\ 0 < y < b, \tag{12}$$

$$v(x,0) = f_1(x), \qquad 0 < x < a, \tag{13}$$

$$v(x,b) = f_2(x), \qquad 0 < x < a, \tag{14}$$

$$v(0,y) = g_1(y), \qquad 0 < y < b, \tag{15}$$

and

$$v(a,y) = g_2(y). \qquad 0 < y < b. \tag{16}$$

We then assume that the solution to problem (6)-(11) can be written as

$$h(x,y,t) = w(x,y,t) + v(x,y) \tag{17}$$

where $w(x,y,t)$ is called the *transient solution*.

In order to find $w(x,y,t)$, we rewrite Eq. (17) as

$$w(x,y,t) = h(x,y,t) - v(x,y). \tag{18}$$

Then, differentiating both sides of (18) with respect to x twice, y twice, and t once,

$$w_{xx}(x,y,t) = h_{xx}(x,y,t) - v_{xx}(x,y), \tag{19}$$

$$w_{yy}(x,y,t) = h_{yy}(x,y,t) - v_{yy}(x,y), \tag{20}$$

and

$$w_t(x,y,t) = h_t(x,y,t) - v_t(x,y) = h_t(x,y,t). \tag{21}$$

It follows from Eqs. (19) and (20)

$$\begin{aligned} w_{xx} + w_{yy} &= (h_{xx} - v_{xx}) + (h_{yy} - v_{yy}) \\ &= (h_{xx} + h_{yy}) - (v_{xx} + v_{yy}). \end{aligned} \tag{22}$$

Then by Eqs. (6) and (12), Eq. (22) can be written as

$$w_{xx} + w_{yy} = \frac{1}{k} h_t - 0 = \frac{1}{k} h_t \tag{23}$$

Also, dividing Eq. (21) by k leads to

$$\frac{1}{k} w_t = \frac{1}{k} h_t. \tag{24}$$

Therefore, Eqs. (23) and (24) show that $w(x,y,t)$ solves the equation

$$w_{xx} + w_{yy} = \frac{1}{k} w_t. \tag{25}$$

Using Eqs. (7)-(11) and (13)-(16), we can find boundary conditions for $w(x,y,t)$:

$$\begin{aligned} w(x,0,t) &= h(x,0,t) - v(x,0) \\ &= f_1(x) - f_1(x) = 0, \qquad 0 < x < a, \end{aligned}$$

$$\begin{aligned} w(x,b,t) &= h(x,b,t) - v(x,b) \\ &= f_2(x) - f_2(x) = 0, \qquad 0 < x < a, \end{aligned}$$

$$\begin{aligned} w(0,y,t) &= h(0,y,t) - v(0,y) \\ &= g_1(y) - g_1(y) = 0, \qquad 0 < y < b, \end{aligned}$$

and

$$\begin{aligned} w(a,y,t) &= h(a,y,t) - v(a,y) \\ &= g_2(y) - g_2(y) = 0, \qquad 0 < y < b, \end{aligned}$$

as well as an initial condition

$$w(x,y,0) = h(x,y,0) - v(x,y)$$

$$= h(x,y,0) - v(x,y) = f(x,y) - v(x,y), \qquad 0 < x < a, \ \ 0 < y < b.$$

Therefore, the transient *w(x,y,t)* is the solution to the linear homogeneous initial value, boundary value problem

$$w_{xx} + w_{yy} = \frac{1}{k} w_t, \tag{26}$$

$$w(x,0,t) = 0 \qquad 0 < x < a, \ \ t > 0, \tag{27}$$

$$w(x,b,t) = 0 \qquad 0 < x < a, \ \ t > 0, \tag{28}$$

$$w(0,y,t) = 0 \qquad 0 < y < b, \ \ t > 0, \tag{29}$$

$$w(a,y,t) = 0 \qquad 0 < y < b, \ \ t > 0, \tag{30}$$

and

$$w(x,y,0) = f(x,y) - v(x,y) \qquad 0 < x < a, \ 0 < y < b. \tag{31}$$

The solution *h(x,y,t)* to the boundary value problem given by Eqs. (6)-(11) is now found by solving the transient and steady state boundary value problems; as stated in Eq. (17), the sum of those solutions will be equal to *h(x,y,t)*.

3.1 Solving the Transient Problem

We begin with the Eq. (26), boundary conditions given by Eqs. (27)-(30), and initial condition given by Eq. (31), which set the transient function equal to zero at the boundary of the rectangular area, and describe the initial value of the transient function at time *t=0* by the function *f(x,y)-v(x,y)*, which we choose to call *g(x,y)*.

To solve problem (26)-(31), we use the technique of *separation of variables* or *Fourier's Method* (Powers,126). This classic technique, which was used by Joseph Fourier to study heat flow in objects, will be the key to solving several problems that appear throughout this thesis. Assuming that the solution to (26) can be written as

$$w(x, y, t) = \phi(x, y)T(t), \tag{32}$$

then substituting *w(x, y)* of form (32) into (26), we find

$$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) T = \frac{1}{k} \phi \frac{\partial T}{\partial t}.$$

Dividing through by $T(t)\phi(x,y)$ to separate variables gives

$$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \frac{1}{\phi} = \frac{1}{kT} \frac{\partial T}{\partial t}. \tag{33}$$

Because the right side of the Eq. (33) is a function of x and y and the left side of the equation is a function of t, in order for this equality to hold for $0 < x < a$, $0 < y < b$, $0 < t$, the common value of the two functions must be some constant, A. This leads to the differential equations

$$T'(t) - AkT(t) = 0, \qquad t > 0 \tag{34}$$

and

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = A\phi(x,y) \qquad 0 < x < a,\ 0 < y < b. \tag{35}$$

The boundary conditions (27)-(30) become

$$\phi(x, 0)T(t) = 0, \qquad 0 < x < a, \quad t > 0, \tag{36}$$

$$\phi(x, b)T(t) = 0, \qquad 0 < x < a, \quad t > 0, \tag{37}$$

$$\phi(0, y)T(t) = 0, \qquad 0 < y < b, \quad t > 0, \tag{38}$$

and

$$\phi(a, y)T(t) = 0, \qquad 0 < y < b, \quad t > 0. \tag{39}$$

In order for all four of Eqs. (36)-(39) to hold, either T(t) = 0 for every t, or $\phi = 0$ on each boundary. Choosing T(t) = 0 leads to the trivial solution, $w(x,y,t) = 0$ for all x, y, and t, so we choose $\phi = 0$ for each case, which gives the following boundary conditions for ϕ:

$$\phi(x, 0) = 0, \qquad 0 < x < a, \tag{40}$$

$$\phi(x, b) = 0, \qquad 0 < x < a, \tag{41}$$

$$\phi(0, y) = 0, \qquad 0 < y < b, \tag{42}$$

and

$$\phi(a, y) = 0, \qquad 0 < y < b. \tag{43}$$

Thus, ϕ solves a new boundary value problem, namely (35), (40)-(43), for which we will again use separation of variables to find ϕ, assuming

$$\phi(x, y) = X(x)Y(y). \tag{44}$$

With assumption (44) applied to (35), after dividing both sides by $\phi(x,y) = X(x)Y(y)$, (35) becomes

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = A \tag{45}$$

Because the sum of a function of x and a function of y is a constant, the two quotients on the left side of (45) must also be constant; i.e.

$$\frac{X''(x)}{X(x)} = B \tag{46}$$

and

$$\frac{Y''(y)}{Y(y)} = C, \tag{47}$$

for some constants B and C.

From (46) and (47), it follows that X and Y must satisfy

$$X''(x) - BX(x) = 0, \qquad 0 < x < a, \tag{48}$$

and

$$Y''(y) - CY(y) = 0, \qquad 0 < y < b. \tag{49}$$

Also, using (44), the boundary conditions (40)-(43) become

$$X(x)Y(0) = 0, \qquad 0 < x < a, \tag{50}$$

$$X(x)Y(b) = 0, \qquad 0 < x < a, \tag{51}$$

$$X(0)Y(y) = 0, \qquad 0 < y < b, \tag{52}$$

and

$$X(a)Y(y) = 0, \qquad 0 < y < b. \tag{53}$$

Again, in order to keep ϕ from being equal to zero for all x and y, neither X or Y can be zero for all x nor for all y respectively, so we require each of the functions X and Y to be zero at the endpoints of their intervals of definition:

$$Y(0) = 0, \tag{54}$$

$$Y(b) = 0, \tag{55}$$

$$X(0) = 0, \tag{56}$$

and

$$X(a) = 0. \tag{57}$$

We now have two independent problems in which we have a homogeneous ordinary differential equation with homogeneous boundary conditions: Eqs. (48), (56)-(57) are one problem and Eqs. (49), (54)-(55) form the other. Such problems are called *eigenvalue* problems (Powers, 133).

To find the solutions to the eigenvalue problem (49), (54)-(55), called the *eigenfunctions*, we use the *characteristic equation method* (Boyce and Diprima, 138), which transforms Eq. (49) into the polynomial

$$m^2 - C = 0$$

which has roots

$$m = \pm\sqrt{C},$$

assuming $C > 0$.

Then it follows that the general solution to Eq. (49) is

$$Y(y) = c_1 \cosh y\sqrt{C} + c_2 \sinh y\sqrt{C}, \quad (58)$$

where c_1 and c_2 are arbitrary constants.

Applying boundary conditions Eqs. (54) and (55) to (58), we get

$$0 = Y(0) = c_1,$$

and

$$0 = Y(b) = c_2 \sinh(b\sqrt{C}),$$

Since b and C are not zero, $c_2=0$ must hold.

Since $c_1 = c_2 = 0$, Eq. (58) becomes

$$Y(y) \equiv 0,$$

yielding the trivial solution

$$w(x,y,t) \equiv 0,$$

which we don't want.

In order to find a non-trivial solution $Y(y)$, we now repeat the above steps to solve (49), this time assuming $C = 0$. The characteristic polynomial in this case is

$$m^2 = 0,$$

so the general solution to Eq. (49) becomes

$$Y(y) = c_1 + yc_2. \quad (59)$$

Applying boundary conditions Eqs. (54) and (55) to Eq. (59), we get

$$0 = Y(0) = c_1,$$

and

$$0 = Y(b) = bc_2.$$

Because b is not zero, $c_2=0$ must hold, and we again get the trivial solution, so $C=0$ is not a good choice.

Finally, we try $C<0$, which for notational convenience in what follows, we write as $C=-\nu^2$, for $\nu > 0$. In this case, the characteristic polynomial is

$$m^2+\nu^2=0,$$

which has roots

$$m=\pm i\nu,$$

so the general solution to Eq. (49) is

$$Y(y)=c_1\cos y\nu+c_2\sin y\nu. \tag{60}$$

Applying the boundary condition (54) to (60), we get

$$0=Y(0)=c_1.$$

and applying boundary condition (55) to (60), we find that

$$Y(b)=0=c_2\sin b\nu.$$

Now, either $c_2=0$, or $\sin b\nu=0$. If $c_2=0$, we again get the trivial solution, so we choose $\sin b\nu=0$, which occurs when $\nu^2=\left(\frac{n\pi}{b}\right)^2$, for n =...,-3,-2,-1,1,2,3,...

Letting $c_1=1$, we see that for each non-negative integer, n, we have a solution to problem (49), (54)-(55) of the form

$$Y_n(y)=\sin\left(\frac{n\pi y}{b}\right),\quad \text{with}\quad \nu_n^2=\left(\frac{n\pi}{b}\right)^2,\quad n=\dots -3,-2,-1,1,2,3,\dots \tag{61}$$

Using a similar argument, with $B=-\mu^2$, for $\mu>0$, we find the solution to problem (48), (56)-(57) of the form

$$X_m(x)=\sin\left(\frac{m\pi x}{a}\right),\quad \text{with}\quad \mu_m^2=\left(\frac{m\pi}{a}\right)^2,\quad m=\dots -3,-2,-1,1,2,3,\dots \tag{62}$$

Since we now know the values of constants B and C, we also know the value of constant A. Recall from Eqs. (45)-(47) that

$$\begin{aligned} A&=B+C \\ &=-\mu^2+-\nu^2<0 \\ &=-\lambda^2 \text{ for } \lambda>0. \end{aligned} \tag{63}$$

Thus for each pair of non-negative integers m,n, there is a solution of (35), (40)-(43) of the form (44), i.e.

$$\phi_{mn}(x, y) = X_m(x)Y_n(y), \qquad m=...,-3,-2,-1,1,2,3..., \ n=...,-3,-2,-1,1,2,3... \tag{64}$$

with $X_m(x)$ and $Y_n(y)$ given by Eqs. (61) and (62).

For each m,n, the corresponding function $T_{mn}(t)$, the solution of (34), is found by integrating

$$\frac{T'_{mn}(t)}{T_{mn}(t)} = -\lambda^2 k$$

with respect to t, which leads to

$$\ln|T_{mn}| = -\lambda^2 kt + c_1, \tag{65}$$

for arbitrary constant c_1. Applying the exponential function to both sides of (65), we find

$$T_{mn}(t) = c_2 \exp(-\lambda_{mn}^2 kt) \tag{66}$$

where $c_2 = \pm\exp(c_1)$. Again, choosing $c_2=1$, we get, from (32), (44), (61) – (64), and (66), for each pair of indices m, n (m = -3,-2,-1,1, 2, 3, …, n = -3,-2,-1,1, 2, 3, …), a function

$$\begin{aligned} w_{mn}(x, y, t) &= \phi_{mn}(x, y)T_{mn}(t) \\ &= \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\exp(-\lambda_{mn}^2 kt), \end{aligned}$$

which satisfies the partial differential equation (26) and the boundary conditions (27)-(31).

The *Principle of Superposition* (Powers, 4) states that any linear combination of the solutions of a linear homogeneous equation is also a solution of the equation. Since (12) is a linear homogeneous equation, we'd expect that a linear combination of the form

$$w_{mn}(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\phi_{mn}(x, y)T_{mn}(t) \tag{67}$$

may also solve (26)-(31). (Note that this sum in (67) also includes all terms with m and n equal to negative integers, because $\sin(-\theta) = -\sin(\theta)$.)

To show this formally, we substitute Eq. (67) into Eq. (26)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn}T(t)\frac{\partial^2\phi_{mn}(x,y)}{\partial x^2} + a_{mn}T(t)\frac{\partial^2\phi_{mn}(x,y)}{\partial y^2}\right] = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k}\left(a_{mn}\phi_{mn}(x,y)\frac{\partial T(t)}{\partial t}\right);$$

which becomes

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn}e^{-\lambda^2 kt}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left[-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\right]\right]$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k}a_{mn}\left(-\lambda_{mn}^2 ke^{-\lambda^2 kt}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

Cancelling like terms, this reduces to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\right]=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\lambda_{mn}^2)$$

which is true, by the definition of $-\lambda^2$ in (63).

We now show formally that the boundary conditions Eqs. (27)-(31) are satisfied by w_{mn} of form (67), by using the fact that each $\phi_{mn}(x,y)$ solves Eqs. (40)-(43). In particular,

$$w(x,0,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\phi_{mn}(x,0)T_{mn}(t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}*0*T_{mn}(t)=0,$$

$$w(x,b,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\phi_{mn}(x,b)T_{mn}(t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}*0*T_{mn}(t)=0,$$

$$w(0,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\phi_{mn}(0,y)T_{mn}(t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}*0*T_{mn}(t)=0,$$

and

$$w(a,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\phi_{mn}(a,y)T_{mn}(t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}*0*T_{mn}(t)=0.$$

For (67) to be a solution of (26)-(31), the initial condition given by Eq. (31) still has to be satisfied. If h has the above form (67), then the initial condition becomes, substituting Eq. (67) into Eq. (31), with $t=0$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\phi_{mn}(x,y) = g(x,y) \tag{68}$$

Multiplying both sides of Eq. (68) by

$$\phi_{pq}(x,y)=\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{a}\right),$$

for fixed p and q and integrating, we find that formally

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \int_0^b \int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)dxdy$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_0^b \int_0^a g(x,y)\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)dxdy \qquad (69)$$

must hold.

Now, in the case that m, n, p, and q are not equal,

$$\int_0^b \int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)dxdy$$

$$= \frac{ab}{4\pi^2}\left[\left(\frac{\sin\pi(m-p)}{m-p} - \frac{\sin\pi(m+p)}{m+p}\right)\left(\frac{\sin\pi(n-q)}{n-q} - \frac{\sin\pi(n+q)}{n+q}\right)\right]$$

which will always equal zero, because m, n, p, and q are integers and the sine of any integer multiple of π is equal to zero.

For the case when $m=p$, and $n=q$, we get

$$\int_0^b \int_0^a \sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)dxdy = \left(\frac{a}{2} - \frac{a\sin(2m\pi)}{4m\pi}\right)\left(\frac{b}{2} - \frac{b\sin(2n\pi)}{4n\pi}\right).$$

Again, because m and n, are integers, this reduces to

$$\int_0^b \int_0^a \sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)dxdy = \frac{ab}{4}.$$

Therefore

$$\int_0^b\int_0^a \phi_{mn}(x,y)\phi_{pq}(x,y)\,dx\,dy = \begin{cases} \frac{ab}{4} \text{ if } m = p \text{ and } n = q \\ 0, \text{ otherwise.} \end{cases} \qquad (70)$$

We call Eq. (70) an *orthogonality condition* (Powers, 247). Applying (70) to (69), we see that all terms on each side of (69) vanish except ones in which $p = m$ and $q = n$, from which it follows that the formula for the coefficients a_{mn} is

$$a_{mn} = \frac{4}{ab}\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy, \qquad (71)$$

for m=1,2,3,... and n=1,2,3,... Thus, Eq. (67) with coefficients given by Eq. (71) is the solution to the transient initial value-boundary value problem given in Eqs. (26)-(31).

Note that the formal calculations done above can be shown to hold rigorously, but this is beyond the scope of this thesis. The key to this is to show that the series solution given by Eq. (67) converges *uniformly* on appropriate sets (Rudin, 147-154). Also, it can be shown that the *double Fourier series* in (69), with coefficients given by (71) converges to $g(x,y)$, provided the function $g(x,y)$ is sufficiently well-behaved, for example when the first and second partial derivatives of $g(x,y)$ are continuous throughout the rectangle definied by $0<x<a$ and $0<y<b$ (Tolstov, 178 and Asmar, 157-158).

3.2 Solving the Steady State Problem

In order to solve the steady-state boundary value problem (12) – (16), we split it into four separate boundary value problems, for the functions v_1, v_2, v_3, and v_4. Each of these problems will have three homogeneous and one non-homogeneous boundary conditions (Edwards, 636) and is much simpler to solve than the complete steady-state boundary value problem (12)-(16). The solution to the steady-state boundary value problem will then be given by

$$v(x,y) = v_1(x,y) + v_2(x,y) + v_3(x,y) + v_4(x,y). \tag{72}$$

These four boundary value problems are

$$\begin{cases} v_{1xx} + v_{1yy} = 0, & 0<x<a,\ 0<y<b & (73) \\ v_1(0,y) = v_1(a,y) = v_1(x,b) = 0, & 0<x<a,\ 0<y<b & (74) \\ v_1(x,0) = f_1(x), & 0<x<a & (75) \end{cases}$$

$$\begin{cases} v_{2xx} + v_{2yy} = 0, & 0<x<a,\ 0<y<b & (76) \\ v_2(x,0) = v_2(0,y) = v_2(a,y) = 0, & 0<x<a,\ 0<y<b & (77) \\ v_2(x,b) = f_2(x), & 0<x<a,\ 0<y<b & (78) \end{cases}$$

$$\begin{cases} v_{3xx} + v_{3yy} = 0, & 0<x<a,\ 0<y<b & (79) \\ v_3(x,0) = v_3(x,b) = v_3(a,y) = 0, & 0<x<a,\ 0<y<b & (80) \\ v_3(0,y) = g_1(y), & 0<y<b & (81) \end{cases}$$

and

$$\begin{cases} v_{4xx} + v_{4yy} = 0, & 0<x<a,\ 0<y<b & (82) \\ v_4(x,0) = v_4(x,b) = v_4(0,y) = 0, & 0<x<a,\ 0<y<b & (83) \\ v_4(a,y) = g_2(y), & 0<y<b & (84) \end{cases}$$

If we substitute (72) into the original problem in Eqs. (12) – (16) and use the fact that v_1 through v_4 satisfy (73)-(75) through (82)-(84), respectively, we can see that this method will lead to a solution of the original steady-state boundary value problem.

For each of the problems (73)-(75), (76)-(78), (79)-(81), and (82)-(84), we use separation of variables, assuming $v_i(x,y)$ can be written as $v_i(x,y) = X(x)Y(y)$, for i = 1, 2, 3, and 4. Note that these functions X and Y are not the same functions used in the separation of variables solution for the transient problem and will change for each of the problems (73)-(75) to (82)-(84).

Substituting $v(x,y) = X(x)Y(y)$ into the equation Eq. (73) and dividing by $X(x)Y(y)$ gives

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \tag{85}$$

As before, in order for (85) to hold for all values of x and y, both sides of (85) must equal a constant, D. Again, we have three choices for D, D = 0, D > 0, or D < 0. From (85) and (74), to avoid the trivial solution, X and Y must satisfy

$$\begin{cases} X''(x) - DX(x) = 0, & 0 < x < a \quad (86) \\ X(0) = 0, & (87) \\ X(a) = 0, & (88) \end{cases}$$

and

$$\begin{cases} Y''(y) + DY(y) = 0, & 0 < y < b \quad (89) \\ Y(b) = 0, & (90) \\ Y(0) = f_1(x), & (91) \end{cases}$$

The problem given by Eqs. (86)-(88) is exactly the same as one we've already solved, namely (48), (56)-(57) from the transient problem in section 3.1. Thus, we already know that D < 0 must hold and setting D = $-\gamma^2 < 0$ for $\gamma > 0$, we find that for each non-zero integer n, there exists an eigenfunction solution of (86)

$$X_n(x) = \sin(\gamma_n x) \tag{92}$$

with eigenvalue

$$\gamma_n = \frac{n\pi}{a} \tag{93}$$

for $n = \ldots,-3,-2,-1,1,2,3,\ldots$

Then, using the method of characteristic equations (Boyce and DiPrima, 138), for each eigenvalue γ_n given by (93), problem (89) has a general solution of the form

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right), \tag{94}$$

where A_n and B_n are arbitrary constants. Using the boundary condition (90), we see that

$$0 = Y_n(b) = A_n \cosh\left(\frac{n\pi b}{a}\right) + B_n \sinh\left(\frac{n\pi b}{a}\right),$$

so B_n can be written in terms of A_n as

$$B_n = -\frac{A_n \cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}.$$

Thus, $Y_n(y)$ can be written as

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) - \frac{A_n \cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi y}{a}\right)$$

$$= \frac{A_n}{\sinh\left(\frac{n\pi b}{a}\right)} \left[\sinh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) - \cosh\left(\frac{n\pi b}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right]$$

$$= a_n \sinh\left(\frac{n\pi(b-y)}{a}\right). \tag{95}$$

Choosing $a_n=1$, we see that corresponding to each non-zero integer n is a solution to Eqs. (73) and (74) of the form $X(x)Y(y)$, with X_n given by Eqs. (92), (93) and Y_n given by Eq. (95). As before with the transient problem, it follows from the Principle of Superposition, that any linear combination of the form

$$v_1(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y)$$

$$= \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi(b-y)}{a} \tag{96}$$

should be a solution to Eqs. (73) and (74).

To show this formally, we substitute Eq. (96) into Eq. (73) and find

$$v_{1xx} + v_{1yy} = \sum_{n=1}^{\infty} c_n \left[X''(x)Y(y) + X(x)Y''(y)\right]$$

$$= \sum_{n=1}^{\infty} c_n \left[-\left(\frac{n\pi}{a}\right)^2 \sin\frac{n\pi x}{a} \sinh\frac{n\pi(b-y)}{a} + \left(\frac{n\pi}{a}\right)^2 \sinh\frac{n\pi(b-y)}{a} \sin\frac{n\pi x}{a}\right] = 0,$$

and, substituting Eq. (96) into Eq. (74)

$$v_1(0,y) = \sum_{n=1}^{\infty} c_n X(0) Y(y)$$

$$= \sum_{n=1}^{\infty} c_n * 0 * Y(y) = 0,$$

$$v_1(a,y) = \sum_{n=1}^{\infty} c_n X(a) Y(y)$$

$$= \sum_{n=1}^{\infty} c_n * 0 * Y(y) = 0,$$

and

$$v_1(x,b) = \sum_{n=1}^{\infty} c_n X(x) Y(b)$$

$$= \sum_{n=1}^{\infty} c_n * X(x) * Y(b) = 0.$$

To find the coefficients c_n, we use boundary condition (91) with v_1 given by (96) to see that

$$f_1(x) = v_1(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi(b)}{a}. \tag{97}$$

Then, using the same type of idea as in the solution of the transient problem, with the orthogonality condition for integers m and n

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx = \begin{cases} \frac{a}{2} & if\, m = n, \\ 0, & otherwise \end{cases} \tag{98}$$

which is found from the integrals

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{a}{2\pi}\left(\frac{\sin((m-n)\pi x)}{m-n} - \frac{\sin((m+n)\pi x)}{m+n}\right)\Bigg|_0^a = 0$$

when $m \neq n$, and

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx = \left(\frac{x}{2} - \frac{a}{4n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right)\Bigg|_0^a = \frac{a}{2},$$

respectively, we can find the coefficients c_n in (96).

Multiplying (97) by $\sin\left(\frac{m\pi x}{a}\right)$ for fixed integer m on both sides and integrating, it follows formally that

$$\sum_{n=1}^{\infty} c_n \int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}\sinh\frac{n\pi(b)}{a}\,dx = \int_0^a f_1(x)\sin\frac{m\pi x}{a}dx. \tag{99}$$

By Eq. (98) all terms on each side of Eq. (99) vanish except ones in which $m = n$, so

$$c_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f_1(x)\sin\frac{n\pi x}{a}dx,$$

or

$$c_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f_1(x)\sin\frac{n\pi x}{a}dx. \tag{100}$$

Then the solution to problem (73)-(75) is

$$v_1(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a},$$

where the coefficients c_n are given by Eq. (100).

If $f_1(x)$ is periodic function of period 2a that is *sectionally smooth* then the Fourier series given by (96) converges to the average of the right and left limits of $f_1(x)$ at any x-value where $f_1(x)$ is defined. To say that $f_1(x)$ is *sectionally smooth* on a finite interval, we mean that; on this interval, $f_1(x)$ is sectionally continuous, $f'_1(x)$ exists, except possibly at a finite number of points; and $f'_1(x)$ is sectionally continuous. By *sectionally continuous*, we mean that $f_1(x)$ is continuous, except possibly for a finite number of jumps and removable discontinuities (Powers, 60 – 61).

To show rigorously that the series (96) satisfies the differential equation, we use the idea of uniform convergence, which can be shown true for series like (96) if the function $f_1(x)$ is sufficiently well-behaved (Powers, 64-68, 71), the details of which are beyond the scope of this thesis.

The solution to problem (76)-(78) is found in the same way as that for (73)-(75), with $X(x)$ solving (86)-(88) for D = $-\gamma^2$ for $\gamma < 0$ and $Y(y)$ solving (89) with slightly different boundary conditions:

$$\begin{cases} Y''(y) - \gamma^2 Y(y) = 0, \quad 0 < y < b & (101) \\ Y(0) = 0, & (102) \\ Y(b) = f_2(x), & (103) \end{cases}$$

It follows that for each non-zero integer, n, there is a solution $X_n(x)$ given by Eqs. (92) and (93), as before, and Eq. (101) has a corresponding general solution $Y_n(y)$ given by Eq. (94).

Using this solution for $Y_n(y)$ with boundary condition (102), we get a solution to (101)-(102) of the form

$$Y_n(y) = B_n \sinh\frac{n\pi y}{a},$$

for each non-zero integer n, where B_n is an arbitrary constant, which can be taken to be 1, as before.

Then, from the Principle of Superposition, as in the case of problem (73) – (75), it follows that there should be a solution to (76) – (78) of the form

$$v_2(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \tag{104}$$

with coefficients b_n to be determined. To find the coefficients b_n, we use boundary condition (103)

$$f_2(x) = v_2(x,b) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}.$$

Then, by the same process as in the solution of (73)-(75), orthogonality condition (98) implies

$$b_n = \frac{2}{a\sinh\frac{n\pi b}{a}} \int_0^a f_2(x)\sin\frac{n\pi x}{a}\,dx, \tag{105}$$

and the solution to problem (76)-(78) is given by Eq. (104) with the coefficients b_n are by Eq. (105).

The solutions to problems (79)-(81) and (82)-(84) are found similarly. For these problems, we find that D > 0 must hold, so we set D = γ^2, for $\gamma > 0$, and thus $X(x)$ and $Y(y)$ must satisfy

$$X''(x) - \gamma^2 X(x) = 0 \tag{106}$$

and

$$Y''(y) + \gamma^2 Y(y) = 0\,. \tag{107}$$

For both problems (79)-(81) and (82)-(84), $Y(y)$ has the boundary conditions

$$Y(0) = Y(b) = 0, \tag{108}$$

so from the work we did to solve problems (48), (56)-(57) and (86)-(88), problem (107), (108) has eigenvalues and eigenfunctions

$$\gamma_n = \frac{n\pi}{b}, \quad \text{and} \quad Y_n(y) = \sin(\gamma_n y), \quad n = 1,2,3... \tag{109}$$

respectively.

For problem (79)-(81), we also have boundary conditions for $X(x)$:

$$X(a) = 0 \text{ and } X(0) = g_1(y),$$

which means that $X(x)$ solves the same type of problem as (89)-(91), and it follows that the solution to (79)-(81) is

$$v_3(x,y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi(a-x)}{b} \tag{110}$$

where the coefficients d_n are given by

$$d_n = \frac{2}{b\sinh\frac{n\pi a}{b}} \int_0^b g_1(y)\sin\frac{n\pi y}{b}\,dy. \tag{111}$$

The solution to problem (82)-(84) is similar, with eigenfunctions $Y_n(y)$ given by Eq. (109) and corresponding $X_n(x)$ solving (106), with different boundary conditions:

$$X(0) = 0 \text{ and } X(a) = g_2(y),$$

which means that $X_n(x)$ solves a problem similar to the problem (101)-(103). From this, we can see that the solution to (82)-(84) is

$$v_4(x,y) = \sum_{n=1}^{\infty} e_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi x}{b} \tag{112}$$

where the coefficients e_n are given by

$$e_n = \frac{2}{b\sinh\frac{n\pi a}{b}} \int_0^b g_2(y)\sin\frac{n\pi y}{b}\,dy. \tag{113}$$

Formal and rigorous arguments similar to those for (73) – (75) show that the solutions v_2, v_3, and v_4 satisfy problems (76) – (78), (79) –(81), and (82) – (84), respectively.

Now that we've found the solutions to the four problem (73)-(75) through (82)-(84), from (72), the solution to the initial boundary value problem (12)-(16) is a linear combination of the solutions to these four simpler problems

$$v(x,y) = \sum_{n=1}^{\infty} \left[\begin{array}{l} \left(c_n \sinh\frac{n\pi(b-y)}{a} + b_n \sinh\frac{n\pi y}{a} \right) \sin\frac{n\pi x}{a} \\ + \left(d_n \sinh\frac{n\pi(a-x)}{b} + e_n \sinh\frac{n\pi x}{b} \right) \sin\frac{n\pi y}{b} \end{array} \right] \tag{114}$$

where c_n, b_n, d_n, e_n are given by Eqs. (100),(105),(111), and (113), respectively.

3.3 The Complete Solution

Now that we've found the transient solution and the steady-state solution, the solution to the original groundwater flow problem, Eqs. (6)-(11), is given by (17), i.e.

$$h(x,y,t) = w(x,y,t) + v(x,y) \tag{115}$$

where $w(x,y,t)$ is given by Eq. (67), and $v(x,y)$ is given by Eq. (114).

4. An Example of Modeling Groundwater Flow

4.1 Applying the Analytical Model to the Two-Dimensional Groundwater Flow Equation

We now test the analytical solution (25) to the two-dimensional groundwater flow equation on the following example, shown in Figure 3. The initial head values are known and assumed to be fixed for all times at each of the wells on the boundary of the rectangular region. We also assume that the initial head values of the 25 inner wells labeled h_1 through h_{25} are given and can be used to give the initial condition of the problem.

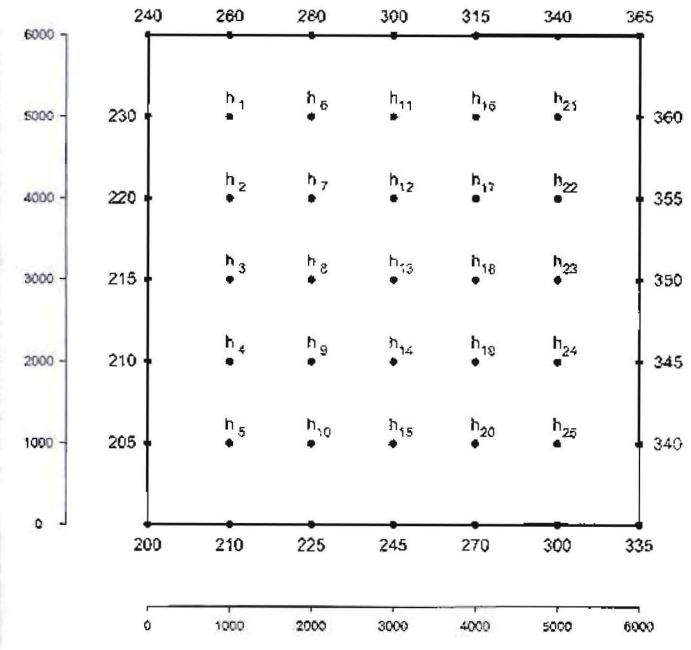

Figure 3: Region of unknown hydraulic head. The rectangular grid units and head heights are measured in feet. (Hadlock, 209).

To find a model for the solution to the steady state problem (12) – (16), we first need to find functions that give the head values at every point on the boundary. This can be done by numerically fitting functions to the known head values at the boundaries. We do this by using the Fit command in Mathematica, which gives us the function that best fits our data, via a least – squares fit, according to the parameters we enter. In this case, we guess that the boundary data would be best fit by quadratic polynomials. These functions become our $f_1(x)$, $f_2(x)$, $g_1(y)$, and $g_2(x)$ in Eqs. (13) – (16). We then use Eq. (114), with the corresponding equations (100),(105),(111), and (113) for finding the Fourier coefficients, c_n, b_n, d_n, and e_n, respectively, to find the steady state value at each well. The constants, a and b are set to 6000, based on Figure 3. We choose to only use terms in our series for n = 1-5, because the coefficients An, Bn, Cn, and Gn are fairly small after that, and it speeds up our calculations. Our steady-state model results are shown in Table 1 and full details of the solution are given in Appendix 1.

Table 1. Steady State values found by applying the analytical method in Mathematica. (Appendix 1)

Well	Steady State Head Value (ft.)
h1	251.089
h2	243.286
h3	235.841
h4	228.402
h5	220.337
h6	271.579
h7	264.331
h8	256.715
h9	248.277
h10	238.243
h11	292.436
h12	285.804
h13	278.401
h14	269.757
h15	259.13
h16	314.029
h17	308.075
h18	301.256
h19	293.211
h20	283.276
h21	336.546
h22	331.306
h23	325.363
h24	318.6
h25	310.555

Next, we look at the transient problem (26) – (31). To test our analytic solution, we choose the function $f(x,y) = xy$ as our initial condition (11). Another way to choose an initial condition would be to use Mathematica's Fit command, to fit given initial data at the 25 inner wells, if this information were available. Then we subtract steady state $v(x,y)$ from $f(x,y)$ to get the initial condition for the transient problem, $g(x,y)$. Then the transient problem is solved by using Eq. (67), with the corresponding equation (71) for the coefficients a_{mn}. We choose a k value of 10,000 ft/day, which is in the range of k values for gravel, through which water easily flows. The numerical calculation times in Mathematica are shortened by using a smaller number of terms for $w(x,y,t)$; we see that after $n = 5$ and $m = 5$, the coefficients H[m,n] become very small, so we choose not to use them, because they are not likely to contribute much to the series solution for *w(x,y,t),* and we can speed up the calculation time by including fewer terms. Finally, a model for the original problem, Eqs. (6)-(11), is then $h(x,y,t)=w(x,y,t)+v(x,y)$. These calculations as well as model output can be found in Appendices 1 and 2.

4.2 Numerical Solution to the Steady-State Problem

Another method of solving the steady-state problem is to use a numerical method, which is especially useful if the points of interest are on a grid, such as in the example shown in Figure 3 above.

Again, for this problem, we know the head values at certain points along the boundary of the region. We may or may not know the initial head values of the wells inside the region; if we don't know them, we can estimate them.

To solve this problem numerically, we set a head value, either known or estimated, for each well, h_i, for i=1,2...25, at time t = 0. Then for each h_i, the head value at a later time t = 1 is found by averaging the head values of the four nearest wells. This process is repeated for later times t = 2, 3... until the head value for each well no longer shows a change from one time step to the next. This gives the steady state head value for each well.

This method is based on the approximation of derivatives of $h(x,y)$. We know, by the *central difference formula* (Hadlock, 207), that the second partial derivatives of the head function at the point (x,y) in Figure 2 can be approximated by head values at the four nearby midpoints of the small rectangle's sides as follows:

$$h_{xx}(x,y) \approx \frac{h(x+\Delta x,y)-2h(x,y)+h(x-\Delta x,y)}{(\Delta x)^2},$$

and

$$h_{yy}(x,y) \approx \frac{h(x,y+\Delta y)-2h(x,y)+h(x,y-\Delta y)}{(\Delta y)^2}.$$

For the steady-state solution, we know that $h(x,y)$ satisfies (7), i.e.

$$h_{xx}+h_{yy}=0,$$

so we can write this last equation as

$$\frac{h(x+\Delta x,y)-2h(x,y)+h(x-\Delta x,y)}{(\Delta x)^2}+\frac{h(x,y+\Delta y)-2h(x,y)+h(x,y-\Delta y)}{(\Delta y)^2}=0.$$

Thus, for the head value of any given well, h_{ij}, in Figure 3, with i = 1000, 2000, ..., 6000 and j = 1000, 2000, ... , 6000, representing points along the x – and y – axis, respectively,

$$\frac{h_{i+1,j}-2h_{i,j}+h_{i-1,j}}{(\Delta x)^2}+\frac{h_{i,j+1}-2h_{i,j}+h_{i,j-1}}{(\Delta y)^2}=0.$$

In this case, we know that Δx and Δy are equal, so we can rewrite the above equation as

$$\frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j} + h_{i,j+1} - 2h_{i,j} + h_{i,j-1}}{(\Delta x)^2} = 0,$$

which can be solved for h_{ij}, giving

$$h_{i,j} = \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}}{4},$$

which is the average of the four surrounding wells.

To illustrate this technique, we use Microsoft Excel to solve the steady-state problem for the example given by Figure 3 above. The boundary data are given, and assumed fixed for all time, but no data is given for the wells inside the region, so we estimate them by guessing, based on the knowledge of the head values of the surrounding wells. A portion of our Excel work is shown in Table 2. For all of the work, see Appendix 5.

Table 2: Steady-state values found from the numerical method.

Well	Initial head value (t=0) (ft.)	Head value (ft.) at t=1	...	Head value (ft.) at t=118	Head value (ft.) at t=119	Head value (ft.) at t=120
h1	250	=(230+260+B7+B3)/4	...	251.413024	251.413024	251.413024
h2	240	=(B2+B8+B4+220)/4	...	243.418948	243.418948	243.418949
h3	230	=(B3+B5+B9+220)/4	...	237.508983	237.508983	237.508983
h4	210	=(B4+B10+B6+210)/4	...	229.219453	229.219453	229.219454
h5	210	=(B5+B11+205+210)/4	...	220.647241	220.647241	220.647241
h6	250	=(280+B2+B12+B8)/4	...	272.233148	272.233148	272.233148
h7	240	=(B3+B7+B9+B13)/4	...	264.753786	264.753787	264.753787
h8	230	=(B4+B8+B10+B14)/4	...	257.397532	257.397532	257.397532
h9	210	=(B5+B9+B11+B15)/4	...	248.721589	248.72159	248.72159
h10	210	=(B6+B10+225+B16)/4	...	238.369512	238.369512	238.369512
h11	270	=(B7+300+B13+B17)/4	...	292.765782	292.765782	292.765782
h12	260	=(B8+B12+B14+B18)/4	...	285.965519	285.965519	285.96552
h13	230	=(B9+B13+B15+B19)/4	...	278.605767	278.605768	278.605768
h14	220	=(B10+B14+B16+B20)/4	...	269.899863	269.899863	269.899863
h15	210	=(B11+B15+245+B21)/4	...	259.109216	259.109217	259.109216
h16	320	=(B12+315+B18+B22)/4	...	312.864461	312.864461	312.864461
h17	260	=(B13+B17+B19+B23)/4	...	307.736741	307.736741	307.736741
h18	240	=(B14+B18+B20+B24)/4	...	301.160158	301.160158	301.160158
h19	220	=(B15+B19+B21+B25)/4	...	293.162877	293.162878	293.162878
h20	260	=(B16+B20+270+B26)/4	...	283.167492	283.167491	283.167492
h21	350	=(B17+340+B23+360)/4	...	335.955322	335.955322	335.955322
h22	350	=(B18+B22+B24+355)/4	...	330.956827	330.956827	330.956827
h23	340	=(B19+B23+B25+350)/4	...	325.135246	325.135246	325.135246
h24	340	=(B20+B24+B26+345)/4	...	318.423999	318.423999	318.423999
h25	335	=(B21+B25+340+300)/4	...	310.397872	310.397873	310.397873

We can see that these steady state head values in the last column of Table 2, at time $t = 120$, are very close, within one or two units, to the values found through the analytical model, which are found in Table 1.

4.3 A combination of the two methods

Because the steady-state solutions found by the two methods are so similar, we can make an adjustment to our analytical model , in order to save some calculation time within Mathematica. To do this, we fit a function, which we call $ss(x,y)$ to the steady-state solution values at the interior wells, which were found either analytically or numerically. We then use this function to find $g(x,y)$ instead of using $v(x,y)$, which contains sums, so $g(x,y)=f(x,y)-ss(x,y)$. The rest of the model is found in the same manner as the previous analytical solution. These calculations can be found in Appendix 3.

5. Discussion and Conclusion

The numerical method works well for wells that are in a grid, but it would be more difficult if the wells were unevenly spaced throughout the region because of the use of nearby wells in the determination of head values, whereas the analytical method results in an equation from which the head value of any point in the region could be found. However, the numerical method is much easier to implement than the analytical method – it is not too hard to set up in a Microsoft Excel spreadsheet, which makes it very easy to run many iterations of averages to reach the steady-state head values.

For either method of finding the steady-state head values, it is not necessary to know the initial head values inside the region – it is not included anywhere in the analytical model, and the numerical method produces the same results regardless of what initial head values are used, although more iterations may be required with some choices.

From the table of graphs in Appendix 2, we can see how the solution $h(x,y,t)$ changes over time. And we can see by the 3D plots below, from Appendix 4, that $h(x,y,0)$ found by the analytical method is very close to the initial function $f(x,y)$, in Figure 4, and $w(x,y,0)$ also found by the analytical method is very close to $g(x,y)$, in Figure 5. We can also see, in Figure 6, that the difference between $ss(x,y)$ and $v(x,y)$ is very small, which makes the substitution of $ss(x,y)$ for $v(x,y)$ possible. Finally we see, in Figures 7-9, that there is little to no difference between $h(x,y,0)$, $h(x,y,1000)$, or $h(x,y,10000)$ based on which method - analytical or combination - we used.

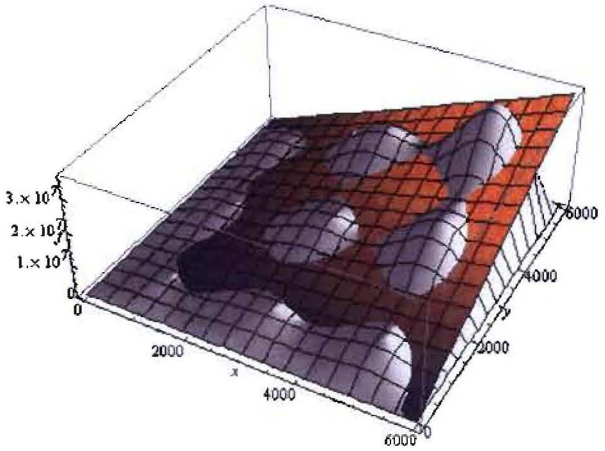

Figure 4: h(x,y,0) in white, and f(x,y) in orange.

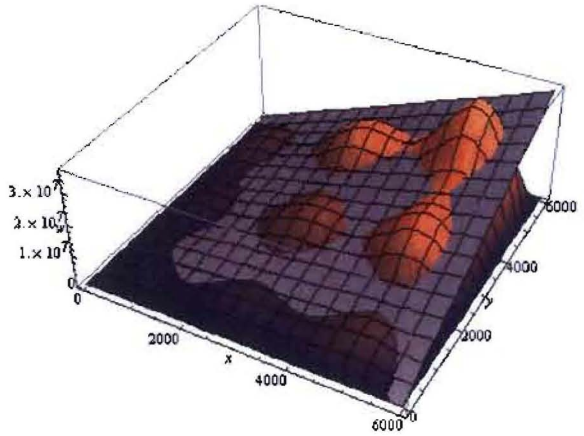

Figure 5: w(x,y,0) in orange and g(x,y) = f(x,y)-v(x,y) in white.

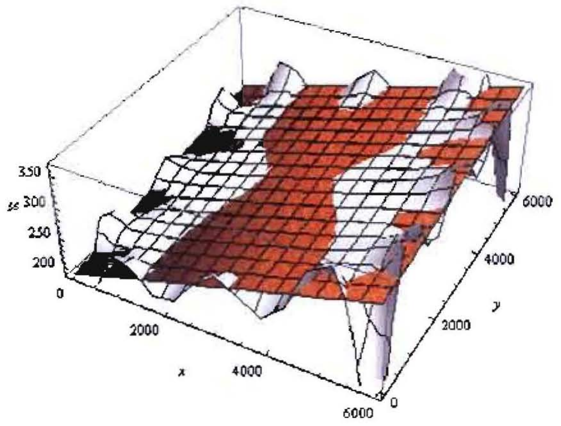

Figure 6: ss(x,y) in orange, and v(x,y) in white.

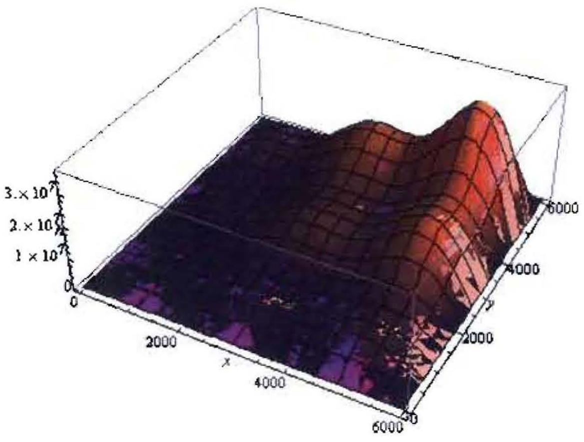

Figure 7: h(x,y,0) found by analytical method (purple) and combination method (orange).

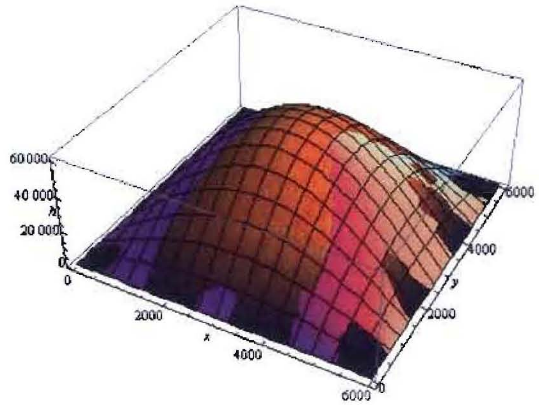

Figure 8: h(x,y,1000) found by analytical method (blue) and combination method (orange).

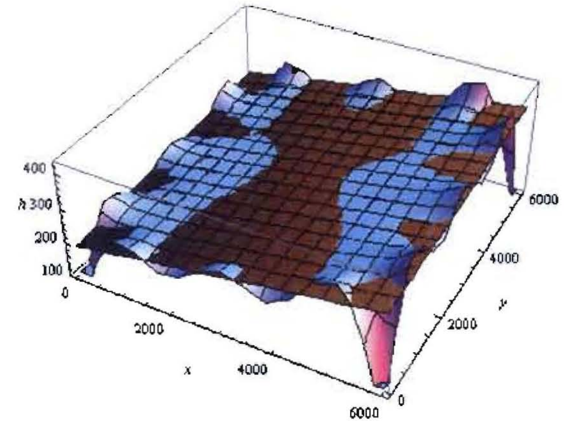

Figure 9: h(x,y,10000) found by analytical method (blue) and combination method (orange).

Using these models, if we know the head heights of points on the boundary of a region and the hydraulic conductivity constant – which can be determined by knowing what kind of soil we have – we can predict the head heights of points anywhere in the region, at any time after initial data is specified. In order to check this, we would need to collect data at each (or many) points in a region, including the boundary, over a long period of time and see if the data matched our predictions.

We did try to use our model on real groundwater data, collected from the USGS website (http://nwis.waterdata.usgs.gov/nwis) but the data found was too complex to be modeled by the model we've developed. A new model, taking changing boundary conditions over time into consideration, needs to be developed to fit that data.

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Appendix 1

Solving the Steady-State Problem

Inputting the known boundary data.

```
In[1]:= bottom = {{0, 0, 200}, {1000, 0, 210}, {2000, 0, 225},
          {3000, 0, 245}, {4000, 0, 270}, {5000, 0, 300}, {6000, 0, 335}}
        top = {{0, 6000, 240}, {1000, 6000, 260}, {2000, 6000, 280}, {3000, 6000, 300},
          {4000, 6000, 315}, {5000, 6000, 340}, {6000, 6000, 365}}
        left = {{0, 0, 200}, {0, 1000, 205}, {0, 2000, 210}, {0, 3000, 215},
          {0, 4000, 220}, {0, 5000, 230}, {0, 6000, 240}}
        right = {{6000, 0, 335}, {6000, 1000, 340}, {6000, 2000, 345},
          {6000, 3000, 350}, {6000, 4000, 355}, {6000, 5000, 360}, {6000, 6000, 365}}

Out[1]= {{0, 0, 200}, {1000, 0, 210}, {2000, 0, 225},
         {3000, 0, 245}, {4000, 0, 270}, {5000, 0, 300}, {6000, 0, 335}}

Out[2]= {{0, 6000, 240}, {1000, 6000, 260}, {2000, 6000, 280},
         {3000, 6000, 300}, {4000, 6000, 315}, {5000, 6000, 340}, {6000, 6000, 365}}

Out[3]= {{0, 0, 200}, {0, 1000, 205}, {0, 2000, 210},
         {0, 3000, 215}, {0, 4000, 220}, {0, 5000, 230}, {0, 6000, 240}}

Out[4]= {{6000, 0, 335}, {6000, 1000, 340}, {6000, 2000, 345},
         {6000, 3000, 350}, {6000, 4000, 355}, {6000, 5000, 360}, {6000, 6000, 365}}
```

Fitting functions to the known boundary data.

```
In[5]:= Fit[bottom, {1, x, x^2}, {x, y}]
        Fit[top, {1, x, x^2}, {x, y}]
        Fit[left, {1, y, y^2}, {x, y}]
        Fit[right, {1, y, y^2}, {x, y}]
```

Out[5]= $200. + 0.0075\,x + 2.5\times10^{-6}\,x^2$

Out[6]= $241.31 + 0.0175\,x + 4.7619\times10^{-7}\,x^2$

Out[7]= $200.833 + 0.00285714\,y + 5.95238\times10^{-7}\,y^2$

Out[8]= $335. + 0.005\,y + 1.4996\times10^{-20}\,y^2$

Naming the functions that define the boundary conditions.

```
In[9]:= f1[x_] := 200 + 0.0075 x + 2.5*^-6 x^2
        f2[x_] := 241.31 + 0.0175 x + 4.7619*^-7 x^2
        g1[y_] := 200.833 + 0.00285714 y + 5.95238*^-7 y^2
        g2[y_] := 335 + 0.005 y + 1.4996*^-20 y^2
```

Setting the values of a and b.

In[13]:= a = 6000
b = 6000

Out[13]= 6000

Out[14]= 6000

Defining the coefficients.

In[15]:=
$$A[n\_] := \frac{2}{a * \mathrm{Sinh}\left[\frac{n*\pi*b}{a}\right]} * \mathrm{NIntegrate}\left[f1[x] * \mathrm{Sin}\left[\frac{n*\pi*x}{a}\right], \{x, 0, a\}\right]$$
$$B[n\_] := \frac{2}{a * \mathrm{Sinh}\left[\frac{n*\pi*b}{a}\right]} * \mathrm{NIntegrate}\left[f2[x] * \mathrm{Sin}\left[\frac{n*\pi*x}{a}\right], \{x, 0, a\}\right]$$
$$F[n\_] := \frac{2}{b * \mathrm{Sinh}\left[\frac{n*\pi*a}{b}\right]} * \mathrm{NIntegrate}\left[g1[y] * \mathrm{Sin}\left[\frac{n*\pi*y}{b}\right], \{y, 0, b\}\right]$$
$$G[n\_] := \frac{2}{b * \mathrm{Sinh}\left[\frac{n*\pi*a}{b}\right]} * \mathrm{NIntegrate}\left[g2[y] * \mathrm{Sin}\left[\frac{n*\pi*y}{b}\right], \{y, 0, b\}\right]$$

Creating tables of the coefficients, to speed up calculations.

In[19]:=
```
An = Table[A[n], {n, 1, 20}]
Bn = Table[B[n], {n, 1, 20}]
Fn = Table[F[n], {n, 1, 20}]
Gn = Table[G[n], {n, 1, 20}]
```

Out[19]= $\{27.481, -0.160495, 0.0181849, -0.000149857, 0.0000204751, -1.86567\times10^{-7}, 2.73481\times10^{-8}, -2.61302\times10^{-10}, 3.97437\times10^{-11}, -3.90374\times10^{-13}, 6.07416\times10^{-14}, -6.075\times10^{-16}, 9.59959\times10^{-17}, -9.72404\times10^{-19}, 1.5538\times10^{-19}, -1.58892\times10^{-21}, 2.56044\times10^{-22}, -2.63753\times10^{-24}, 4.27836\times10^{-25}, -4.43289\times10^{-27}\}$

Out[20]= $\{32.9543, -0.14521, 0.0206867, -0.000135585, 0.0000231976, -1.68799\times10^{-7}, 3.095\times10^{-8}, -2.36416\times10^{-10}, 4.49577\times10^{-11}, -3.53195\times10^{-13}, 6.86944\times10^{-14}, -5.49643\times10^{-16}, 1.0855\times10^{-16}, -8.79794\times10^{-19}, 1.75686\times10^{-19}, -1.43759\times10^{-21}, 2.89488\times10^{-22}, -2.38633\times10^{-24}, 4.83701\times10^{-25}, -4.01071\times10^{-27}\}$

Out[21]= $\{23.7892, -0.0458558, 0.0150451, -0.0000428164, 0.0000168812, -5.33048\times10^{-8}, 2.25263\times10^{-8}, -7.46577\times10^{-11}, 3.27236\times10^{-11}, -1.11535\times10^{-13}, 5.00027\times10^{-14}, -1.73571\times10^{-16}, 7.90152\times10^{-17}, -2.7783\times10^{-19}, 1.27886\times10^{-19}, -4.53977\times10^{-22}, 2.10727\times10^{-22}, -7.53579\times10^{-25}, 3.52102\times10^{-25}, -1.26654\times10^{-27}\}$

Out[22]= $\{38.5872, -0.0356657, 0.023975, -0.0000333017, 0.0000268631, -4.14593\times10^{-8}, 3.58324\times10^{-8}, -5.80672\times10^{-11}, 5.20449\times10^{-11}, -8.67497\times10^{-14}, 7.95199\times10^{-14}, -1.35\times10^{-16}, 1.25653\times10^{-16}, -2.1609\times10^{-19}, 2.03363\times10^{-19}, -3.53094\times10^{-22}, 3.3509\times10^{-22}, -5.86117\times10^{-25}, 5.59891\times10^{-25}, -9.85086\times10^{-28}\}$

Defining v(x,y), using the tables created above.

In[23]:=
$$v[x\_, y\_] := \sum_{n=1}^{5}\left(\left(\mathrm{An}[[n]]\,\mathrm{Sinh}\left[\frac{1}{a}\,n*\pi*(b-y)\right] + \mathrm{Bn}[[n]]\,\mathrm{Sinh}\left[\frac{n*\pi*y}{a}\right]\right) * \mathrm{Sin}\left[\frac{n*\pi*x}{a}\right] + \left(\mathrm{Fn}[[n]]\,\mathrm{Sinh}\left[\frac{1}{b}\,n*\pi*(a-x)\right] + \mathrm{Gn}[[n]]\,\mathrm{Sinh}\left[\frac{n*\pi*x}{b}\right]\right) * \mathrm{Sin}\left[\frac{n*\pi*y}{b}\right]\right)$$

Checking the values of v(x,y) at each of the 25 wells.

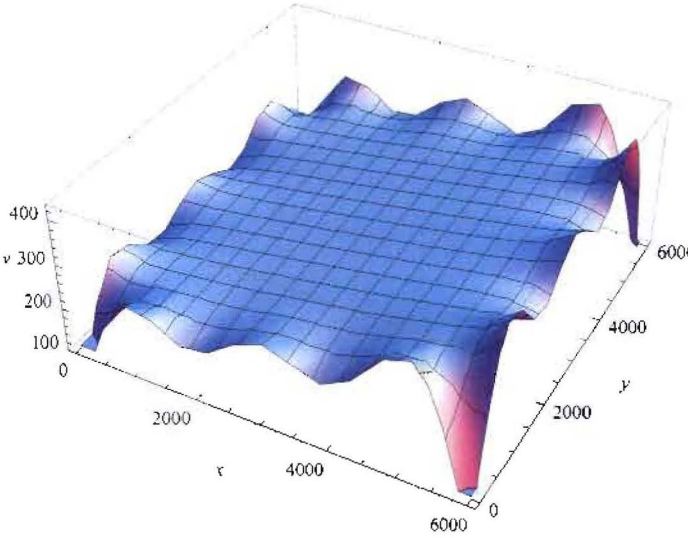
```
In[24]:= V = {v[1000, 5000], v[2000, 5000], v[3000, 5000], v[4000, 5000], v[5000, 5000],
   v[1000, 4000], v[2000, 4000], v[3000, 4000], v[4000, 4000], v[5000, 4000],
   v[1000, 3000], v[2000, 3000], v[3000, 3000], v[4000, 3000], v[5000, 3000],
   v[1000, 2000], v[2000, 2000], v[3000, 2000], v[4000, 2000], v[5000, 2000],
   v[1000, 1000], v[2000, 1000], v[3000, 1000], v[4000, 1000], v[5000, 1000]}

Out[24]= {252.857, 270.468, 293.461, 312.743, 339.098, 242.389, 264.278, 285.835,
   308.006, 329.896, 236.586, 256.737, 278.404, 301.292, 326.549, 227.56, 248.229,
   269.784, 293.146, 317.232, 221.858, 237.282, 260.035, 282.122, 312.917}
```

Plotting the steady-state solution, v(x,y).

```
In[25]:= v1 = Plot3D[v[x, y], {x, 0, 6000}, {y, 0, 6000}, AxesLabel → {x, y, v}]
```

Out[25]=



Appendix 2

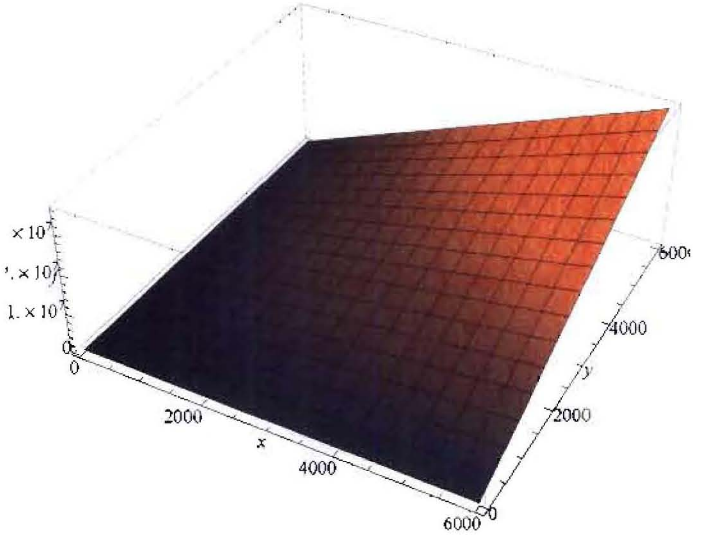
Solving the Transient Problem

Naming the function that defines the initial condition.

```
In[26]:= f[x_, y_] := x * y
```

Plotting the initial condition function.

```
In[27]:= f1 = Plot3D[f[x, y], {x, 0, 6000}, {y, 0, 6000},
          PlotRange → All, AxesLabel → {x, y, f}, ColorFunction → "RustTones"]
```



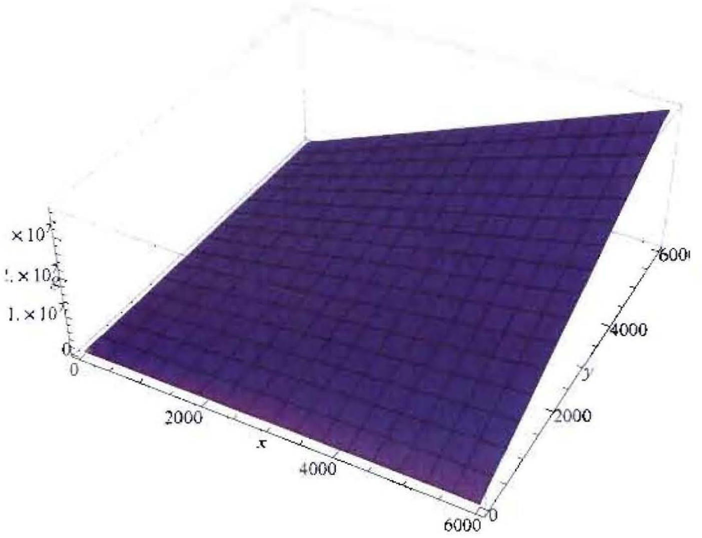
Defining g(x,y) as the difference between f(x,y) and v(x,y). Note that the commands in Appendix 1 for defining steady-state solution v(x,y) must be evaluated before this next command is evaluated.

```
In[28]:= g[x_, y_] := f[x, y] - v[x, y]
```

Plotting g(x,y).

```
g1 = Plot3D[g[x, y], {x, 0, 6000}, {y, 0, 6000}, PlotRange → All, AxesLabel → {x, y, g}]
```

Out[29]=



Setting the value of the constant, k.

```
k = 10 000
```

```
10 000
```

Defining L(m,n).

In[31]:=
```
L[m_, n_] := ((m*π)/a)^2 + ((n*π)/b)^2
```

Defining H(m,n).

In[32]:=
```
H[m_, n_] := 4/(a*b) * NIntegrate[g[x, y] * Sin[(m*π*x)/a] * Sin[(n*π*y)/b], {x, 0, a}, {y, 0, b}]
```

Creating a table of coefficents H(m,n), again, in order to speed up calculations.

```
Hmn = Table[H[m, n], {m, 1, 5}, {n, 1, 5}]
```

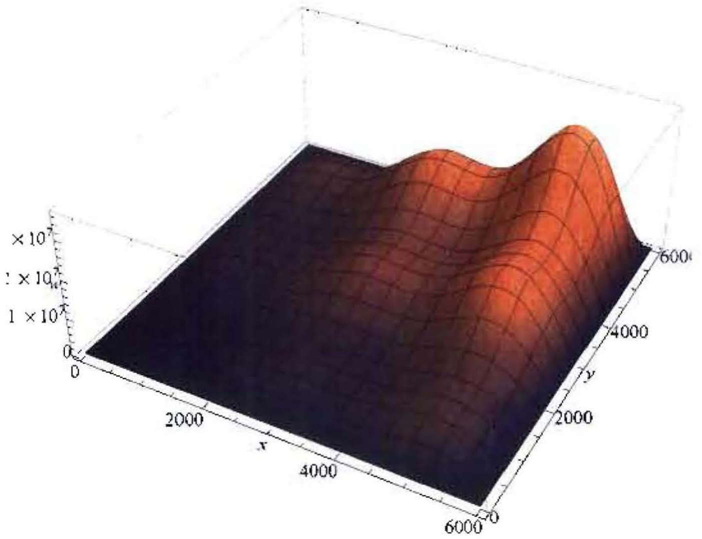
Out[33]=
```
{{1.45898×10^7, -7.29511×10^6, 4.86327×10^6, -3.64755×10^6, 2.91796×10^6},
 {-7.29507×10^6, 3.64756×10^6, -2.43169×10^6, 1.82378×10^6, -1.45901×10^6},
 {4.86326×10^6, -2.4317×10^6, 1.62109×10^6, -1.21585×10^6, 972653.},
 {-3.64754×10^6, 1.82378×10^6, -1.21585×10^6, 911891., -729507.},
 {2.91796×10^6, -1.45902×10^6, 972653., -729511., 583592.}}
```

Defining w(x,y,t).

```
w[x_, y_, t_] := Sum[Sum[(Hmn[[m, n]] * Sin[(m*π*x)/a] * Sin[(n*π*y)/b] * Exp[-L[m, n] * k * t]), {n, 1, 5}], {m, 1, 5}]
```

In[35]:=
```
w1 = Plot3D[w[x, y, 0], {x, 0, 6000},
  {y, 0, 6000}, AxesLabel → {x, y, w}, ColorFunction → "RustTones"]
```

Out[35]=



Solving the 2-Dimensional Groundwater Flow Equation

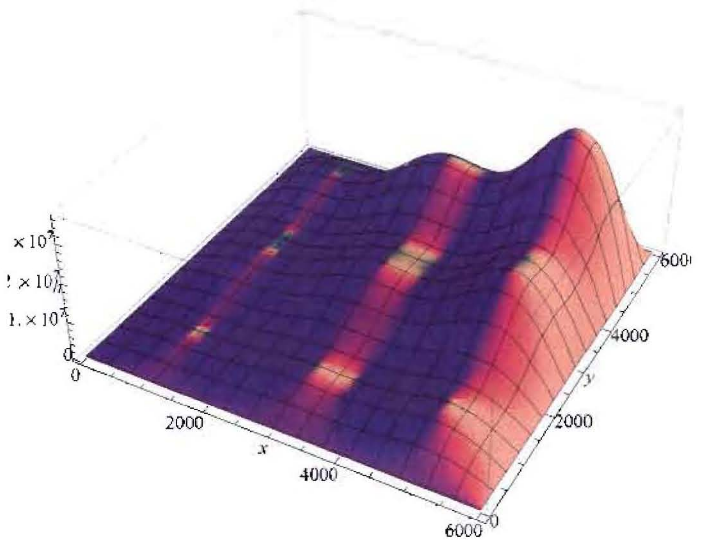
Defining h(x,y,t).

In[36]:=
```
h[x_, y_, t_] := w[x, y, t] + v[x, y]
```

Plotting h(x,y,t) at time t=0.

```
In[37]:= h1 = Plot3D[h[x, y, 0], {x, 0, 6000}, {y, 0, 6000}, AxesLabel → {x, y, h}]
```

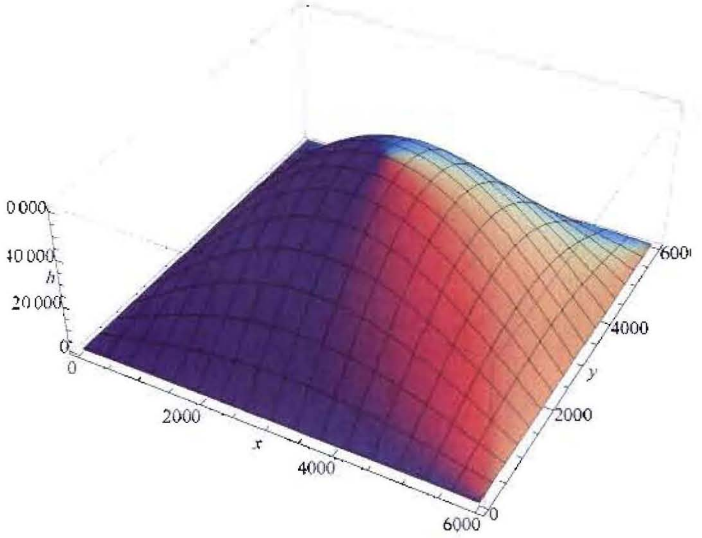
Out[37]=



Plotting h(x,y,t) at time t=1000.

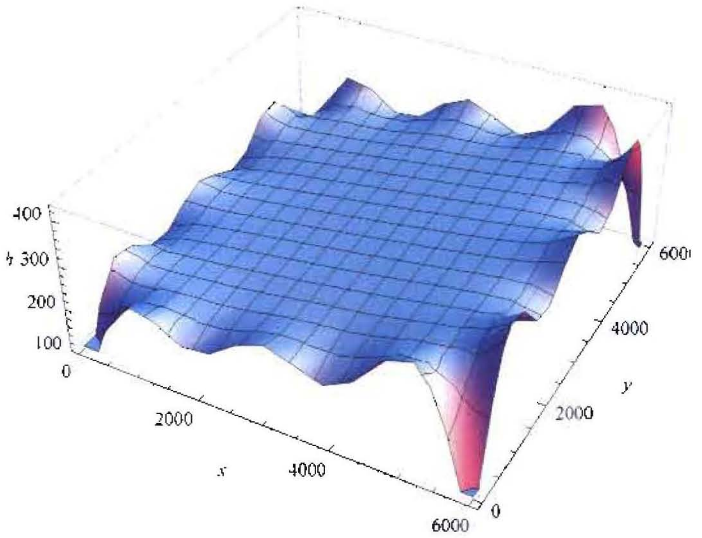
```
In[38]:= h2 = Plot3D[h[x, y, 1000], {x, 0, 6000}, {y, 0, 6000}, AxesLabel → {x, y, h}]
```

Out[38]=



Plotting h(x,y,t) at time t=10,000.

```
h2a = Plot3D[h[x, y, 10000], {x, 0, 6000}, {y, 0, 6000}, AxesLabel → {x, y, h}]
```

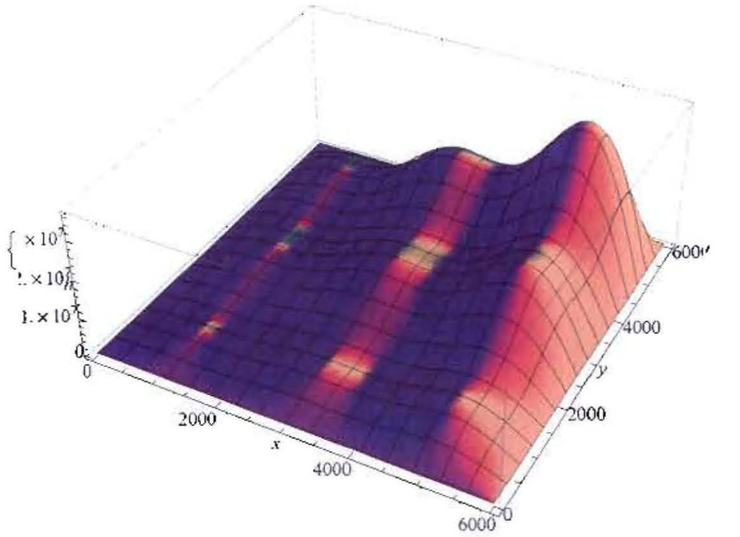


Plotting h(x,y,t) as time increases, to see how it changes.

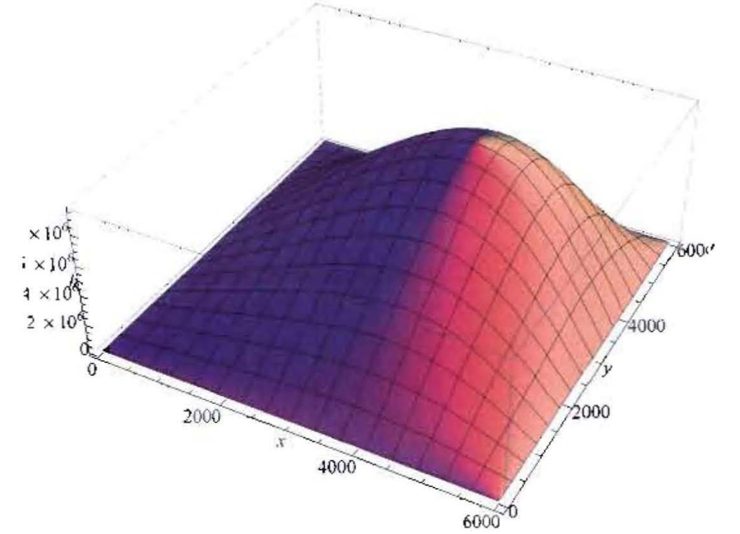
```
Table[Plot3D[Evaluate[h[x, y, t]], {x, 0, a}, {y, 0, b},
  PlotRange → All, AxesLabel → {x, y, h}, PlotLabel → t " =  t"], {t, 0, 2500, 100}]
```

0

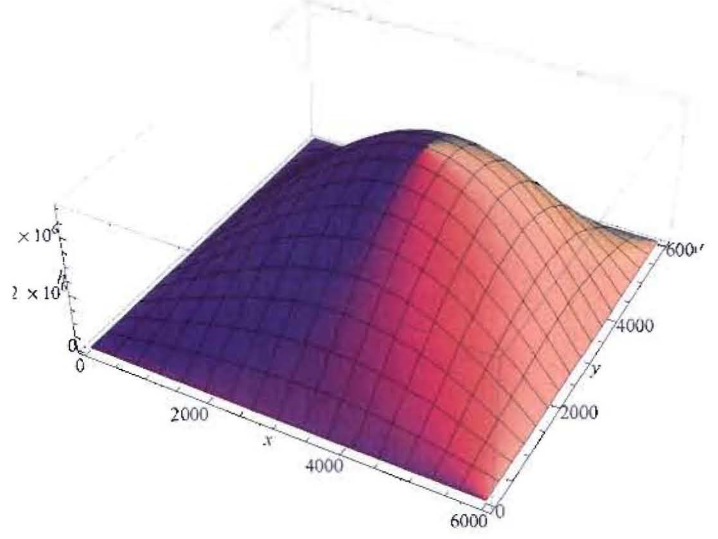
Out[40]=


0
2000
x
4000
6000
0
2000
y
4000
6000

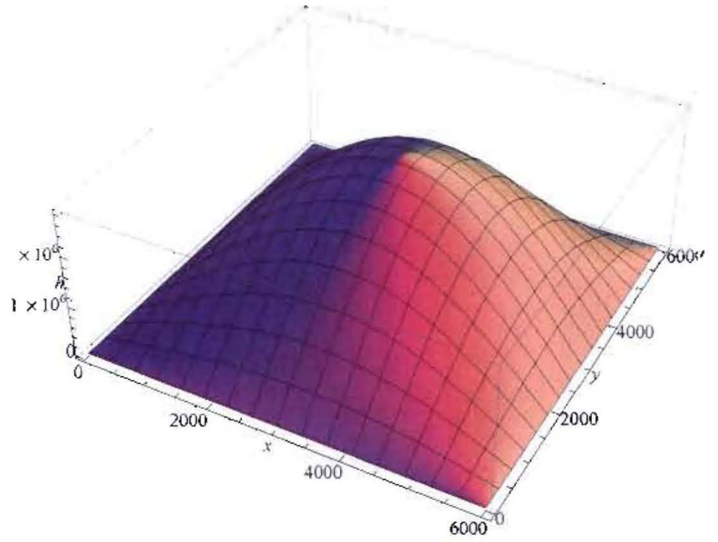
100 = t


0
2 × 10
2000
x
4000
6000
0
2000
y
4000

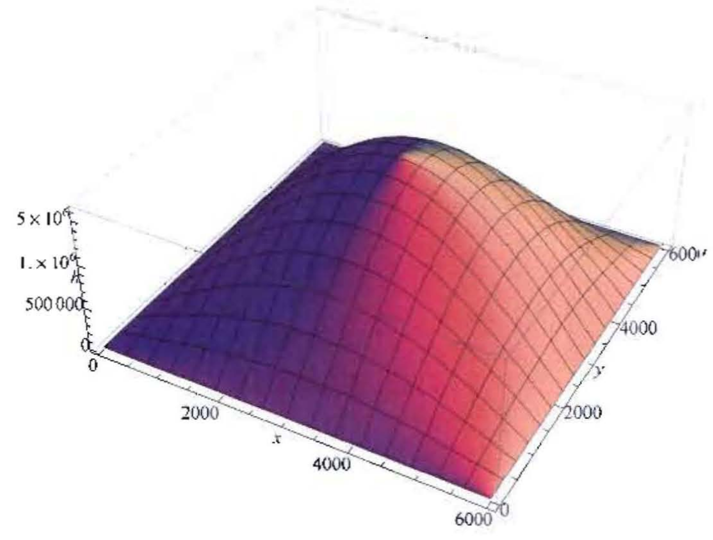
200 = t


2 × 10^6
h
0
2000
4000
6000
x
y
2000
4000

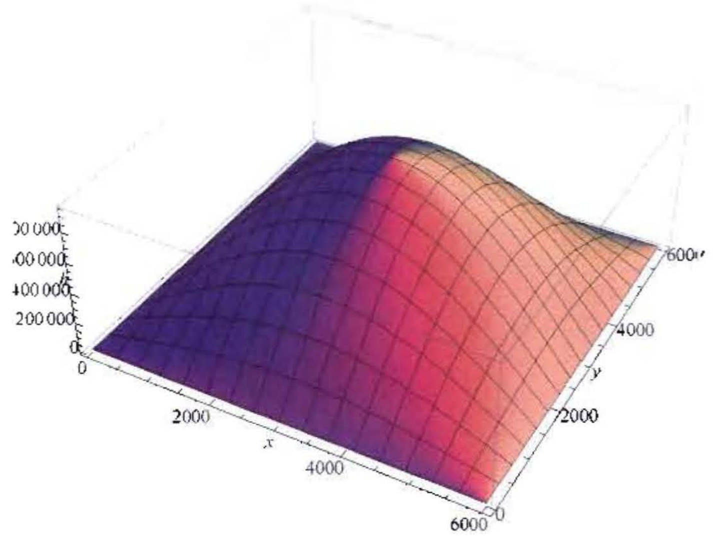
300 = t


1 × 10^6
h
0
2000
4000
6000
x
y
2000
4000

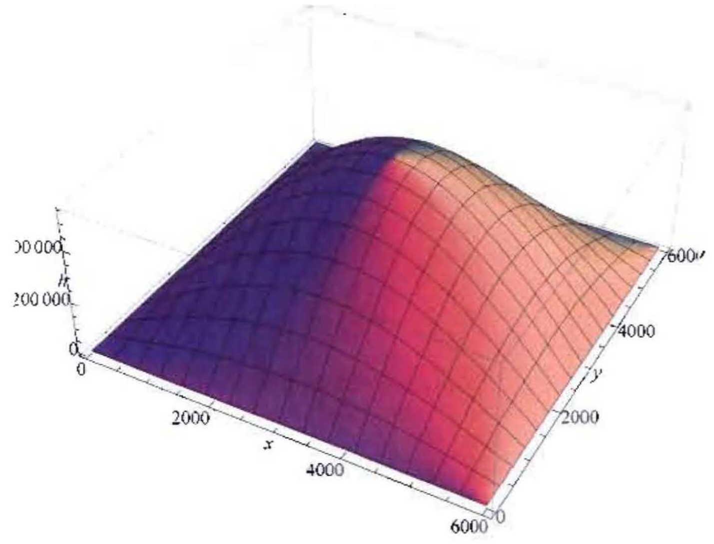
400 = t


5 × 10⁶
1. × 10⁶
500 000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

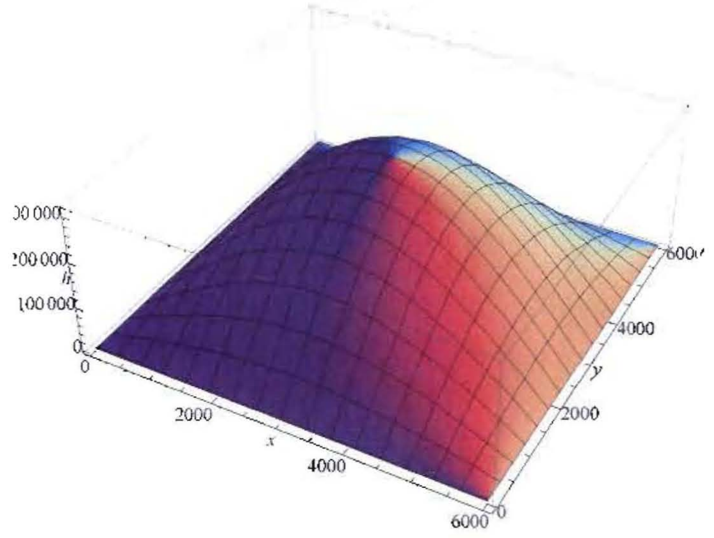
500 = t

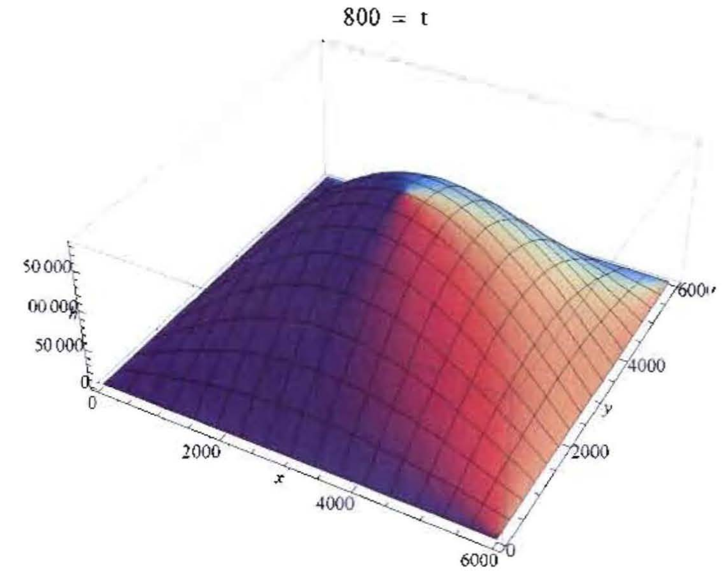

200 000
0
0
2000
x
4000
6000
0
2000
y
4000

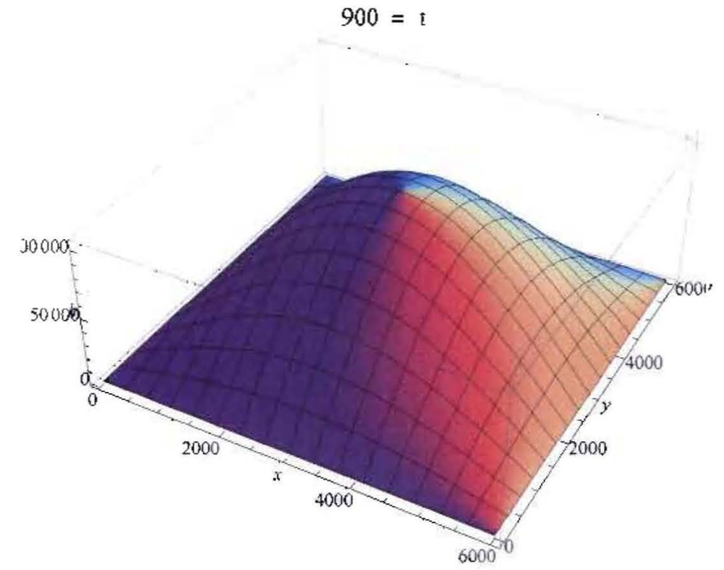
600 = t


00 000
h
200 000
0
0
2000
x
4000
6000
0
2000
y
4000
600

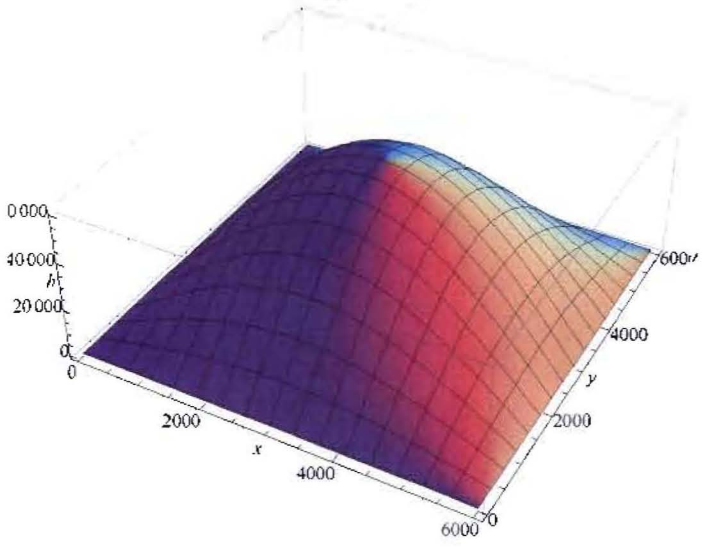
700 = t


00 000
200 000
h
100 000
0
0
2000
x
4000
6000
0
2000
y
4000
600

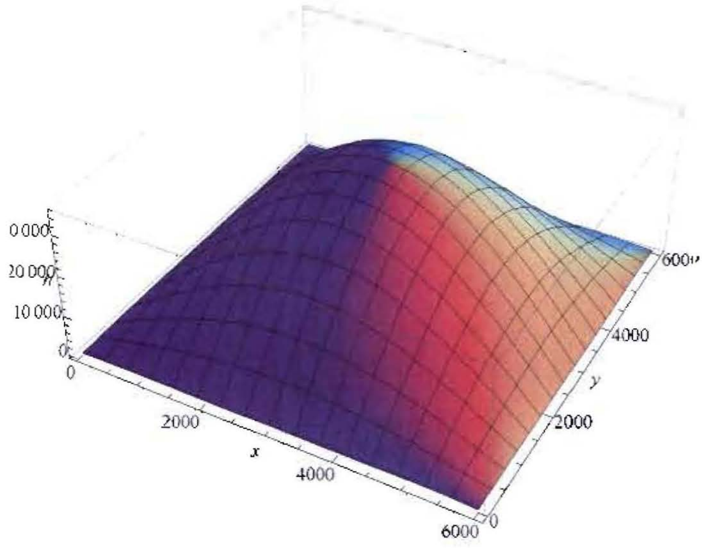

800 = t
50000
00000
50000
0
0
2000
x
4000
6000
0
2000
y
4000
6000


900 = t
50000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

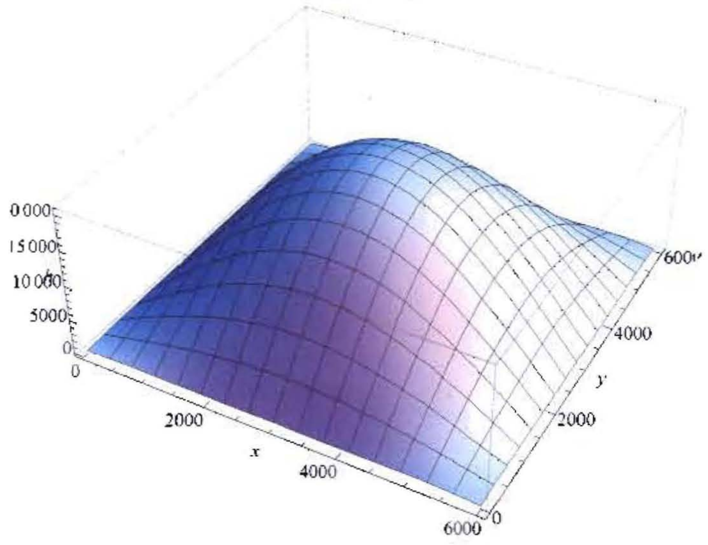
1000 = t


0 000
40 000
h
20 000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

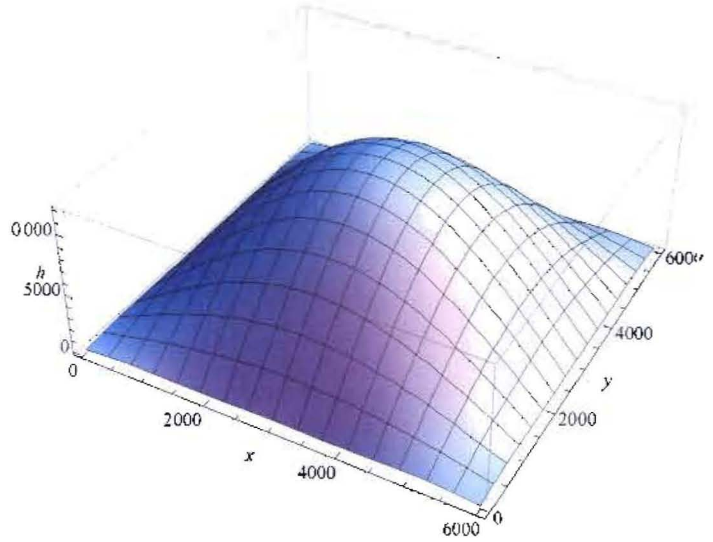
1100 = t


0 000
20 000
h
10 000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

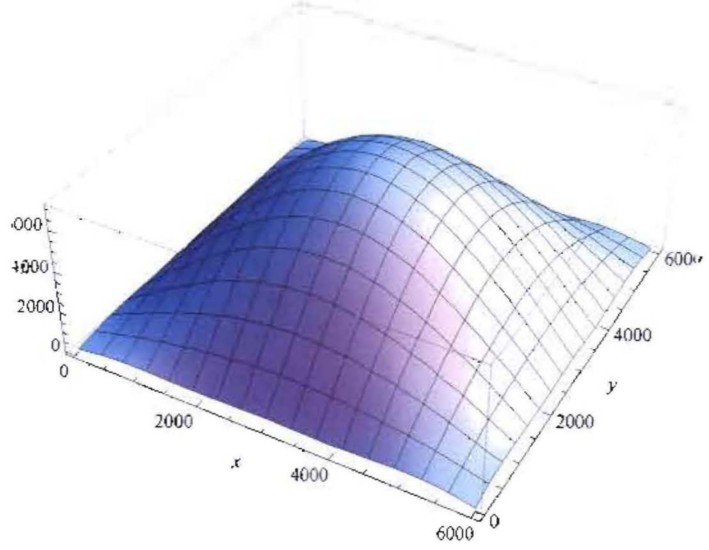
1200 = t


0 000
15 000
h
10 000
5000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

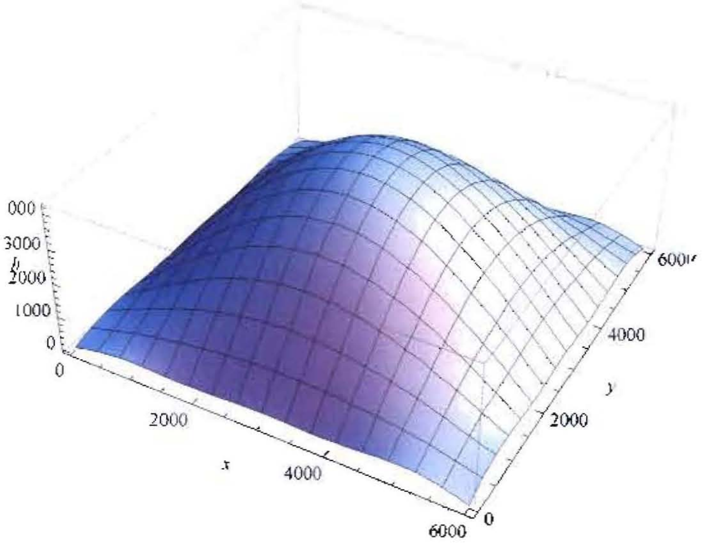
1300 = t


0 000
h
5000
0
0
2000
x
4000
6000
0
2000
y
4000
600

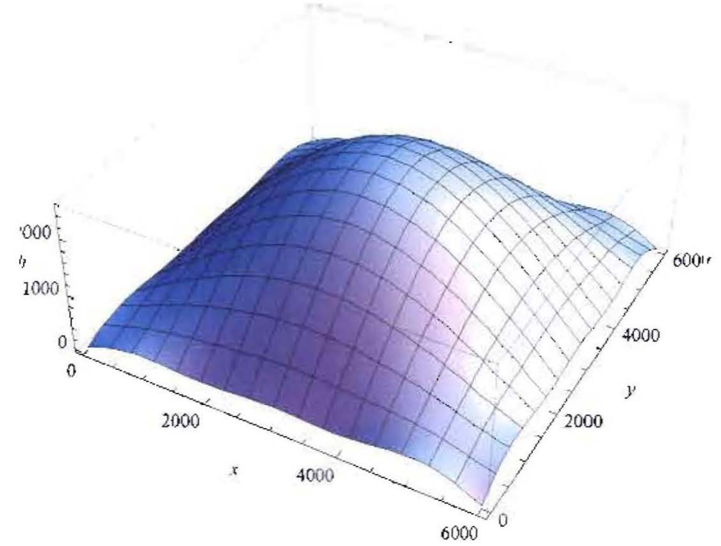
1400 = t


000
4000
2000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

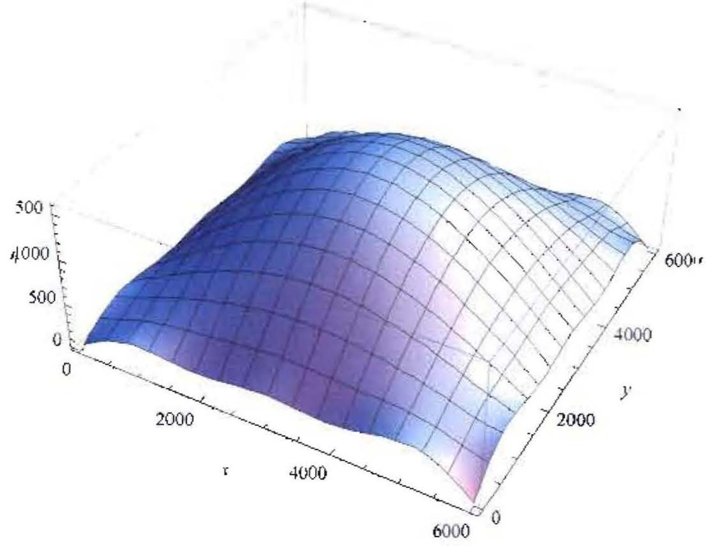
1500 = t

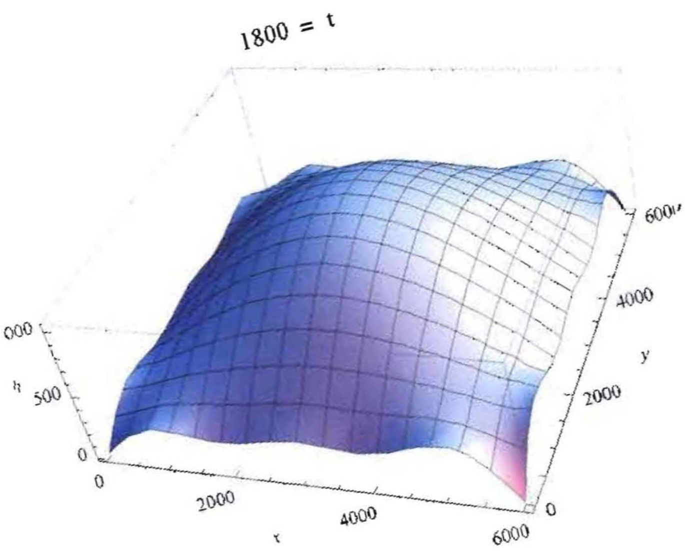

000
3000
h
2000
1000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

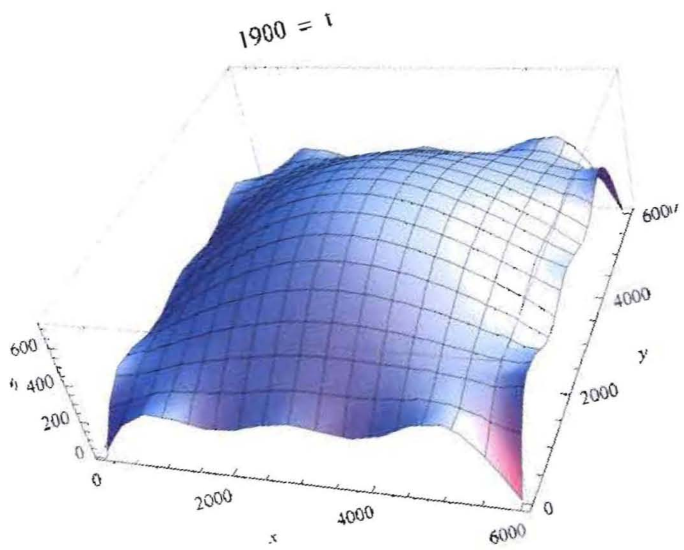
1600 = t

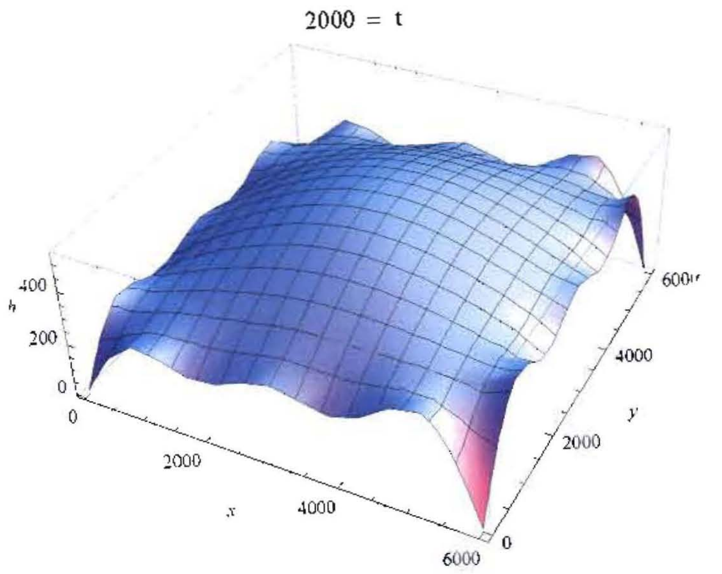

1000
0
0
2000
x
4000
6000
0
2000
y
4000
2000

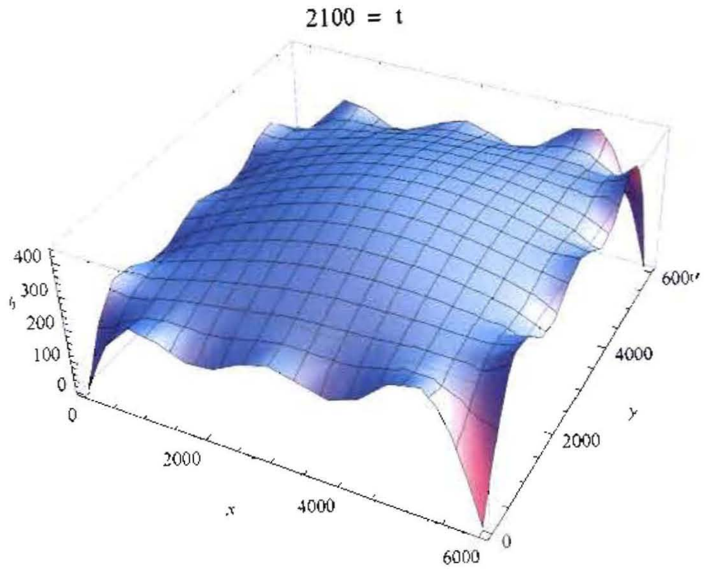
1700 = t


500
0
0
2000
x
4000
6000
0
2000
y
4000

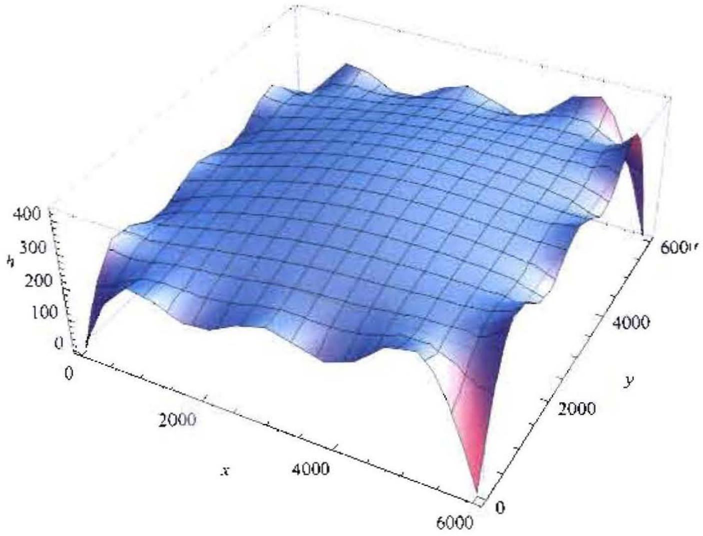

1800 = t
000
h
500
0
0
2000
4000
6000
x
6000
4000
y
2000
0


1900 = t
600
h
400
200
0
0
2000
4000
6000
x
6000
4000
y
2000
0

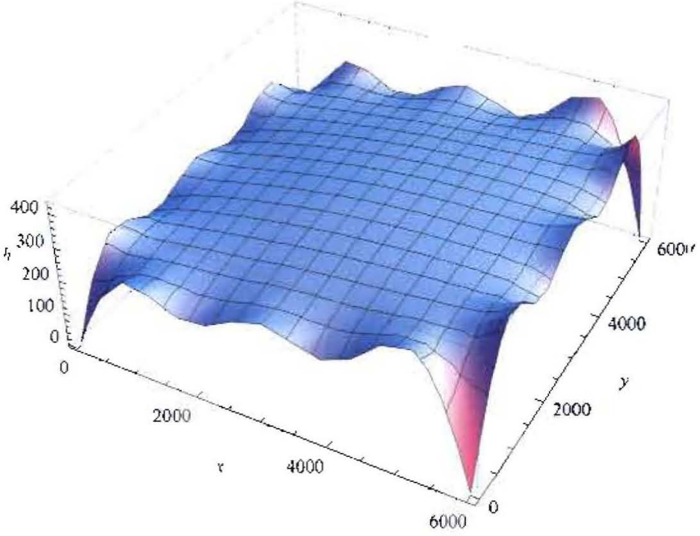

2000 = t
400
h
200
0
0
2000
x
4000
6000
0
2000
y
4000
6000


2100 = t
400
300
h
200
100
0
0
2000
x
4000
6000
0
2000
y
4000
6000

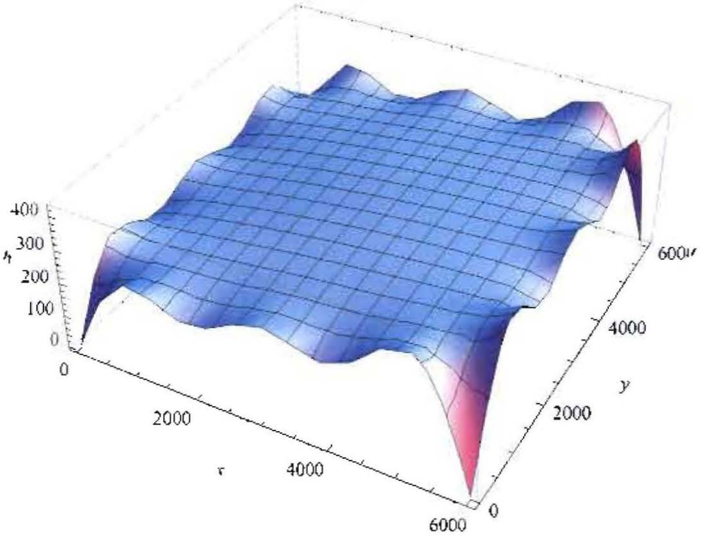
2200 = t



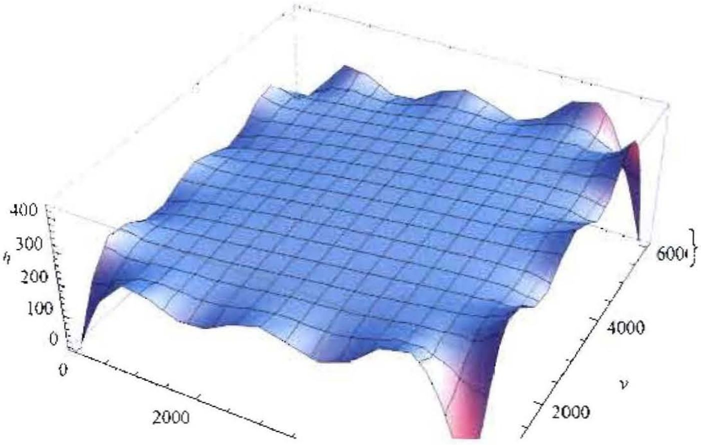
2300 = t



2400 = t



2500 = t



Appendix 3

A variation of the method used to solve transient problem

Inputting our steady state data found above in Appendix 1.

```
In[41]:= steadystate = Partition[
           Flatten[Transpose[Table[{i, j, v[i, j]}, {i, 1000, 5000, 1000}, {j, 5000, 1000, -1000}]]], 3]

Out[41]= {{1000, 5000, 252.857}, {2000, 5000, 270.468}, {3000, 5000, 293.461},
          {4000, 5000, 312.743}, {5000, 5000, 339.098}, {1000, 4000, 242.389},
          {2000, 4000, 264.278}, {3000, 4000, 285.835}, {4000, 4000, 308.006},
          {5000, 4000, 329.896}, {1000, 3000, 236.586}, {2000, 3000, 256.737}, {3000, 3000, 278.404},
          {4000, 3000, 301.292}, {5000, 3000, 326.549}, {1000, 2000, 227.56}, {2000, 2000, 248.229},
          {3000, 2000, 269.784}, {4000, 2000, 293.146}, {5000, 2000, 317.232}, {1000, 1000, 221.858},
          {2000, 1000, 237.282}, {3000, 1000, 260.035}, {4000, 1000, 282.122}, {5000, 1000, 312.917}}
```

Fitting a function to this data.

```
In[42]:= Fit[steadystate, {1, x, y, x*y}, {x, y}]
```

Out[42]= $186.509 + 0.0230815\,x + 0.00856313\,y - 2.99206\times10^{-7}\,x\,y$

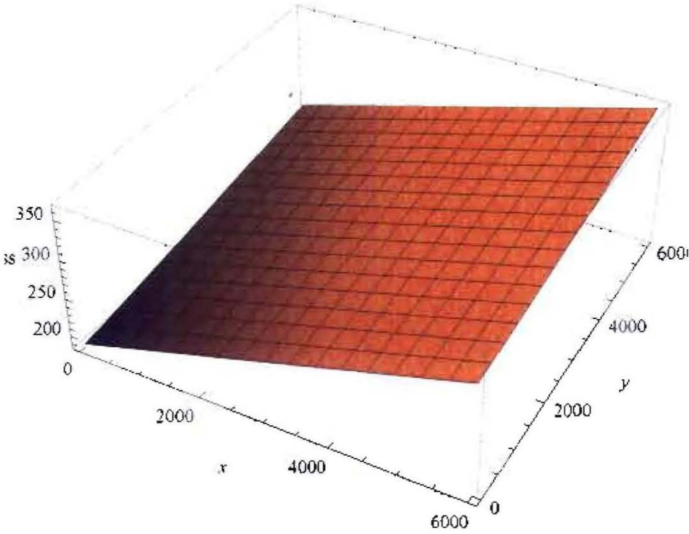
Defining the fuction, ss(x,y), that describes the steady-state values.

```
In[43]:= ss[x_, y_] := 186.5094193298756` + 0.02308154471514651` x +
           0.008563128608128998` y - 2.9920608748511206`*^-7 x y
```

Plotting ss(x,y).

```
In[44]:= s1 = Plot3D[ss[x, y], {x, 0, 6000}, {y, 0, 6000},
           AxesLabel → {x, y, ss}, ColorFunction → "RustTones"]
```

Out[44]=

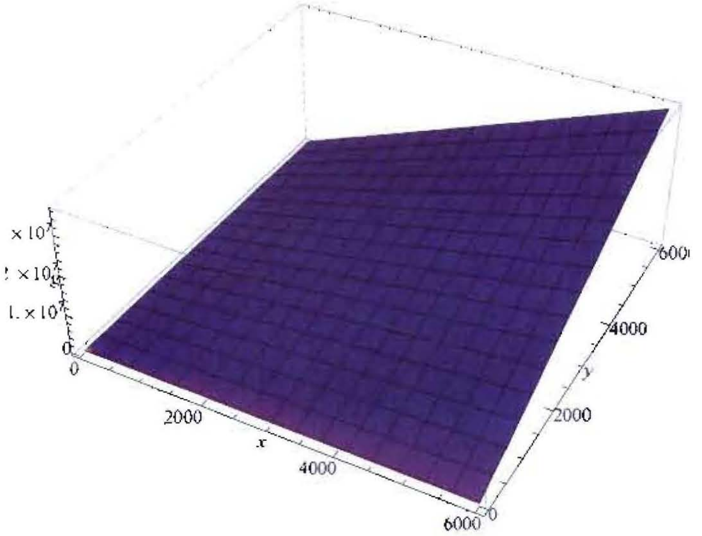


We now repeat the steps performed above to solve the transient problem, except we now define g(x,y) as:

In[45]:= `g[x_, y_] := f[x, y] - ss[x, y]`

In[46]:= `Plot3D[g[x, y], {x, 0, 6000}, {y, 0, 6000}, AxesLabel → {x, y, g}]`

Out[46]=



Setting the value of the constant, k.

In[47]:= `k = 10000`

Out[47]= 10000

Defining L(m,n).

In[48]:= $L[m\_, n\_] := \left(\frac{m*\pi}{a}\right)^2 + \left(\frac{n*\pi}{b}\right)^2$

Defining H(m,n).

In[49]:= $H[m\_, n\_] := \frac{4}{a*b} * \text{NIntegrate}\left[g[x, y] * \text{Sin}\left[\frac{m*\pi*x}{a}\right] * \text{Sin}\left[\frac{n*\pi*y}{b}\right], \{x, 0, a\}, \{y, 0, b\}\right]$

Creating a table of coefficents H(m,n), again, in order to speed up calculations.

In[50]:= `Hmn = Table[H[m, n], {m, 1, 5}, {n, 1, 5}]`

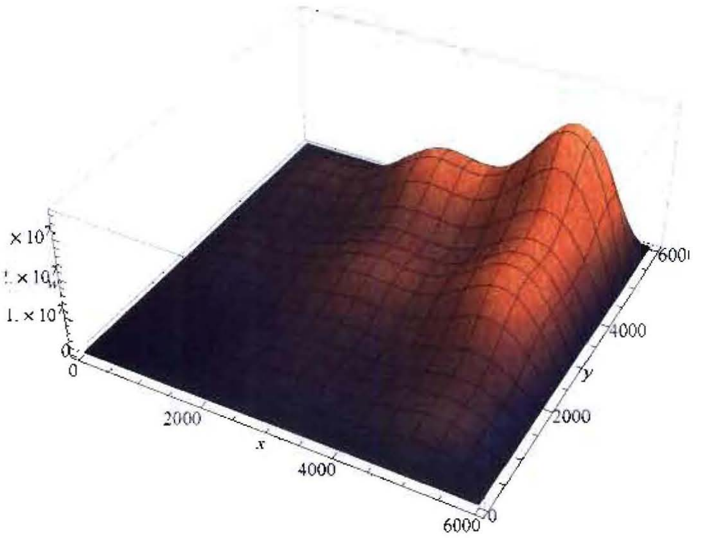
Out[50]= $\{\{1.45898\times10^{7}, -7.29511\times10^{6}, 4.86327\times10^{6}, -3.64755\times10^{6}, 2.91796\times10^{6}\},$
$\{-7.29507\times10^{6}, 3.64756\times10^{6}, -2.43169\times10^{6}, 1.82378\times10^{6}, -1.45901\times10^{6}\},$
$\{4.86327\times10^{6}, -2.4317\times10^{6}, 1.62109\times10^{6}, -1.21585\times10^{6}, 972653.\},$
$\{-3.64754\times10^{6}, 1.82378\times10^{6}, -1.21585\times10^{6}, 911891., -729507.\},$
$\{2.91796\times10^{6}, -1.45902\times10^{6}, 972653., -729511., 583592.\}\}$

Defining w(x,y,t).

In[51]:= $w[x\_, y\_, t\_] := \sum_{m=1}^{5}\sum_{n=1}^{5}\left(\text{Hmn}[[m, n]] * \text{Sin}\left[\frac{m*\pi*x}{a}\right] * \text{Sin}\left[\frac{n*\pi*y}{b}\right] * \text{Exp}[-L[m, n]*k*t]\right)$

In[52]:=
```
Plot3D[w[x, y, 0], {x, 0, 6000}, {y, 0, 6000},
 AxesLabel -> {x, y, w}, ColorFunction -> "RustTones"]
```

Out[52]=



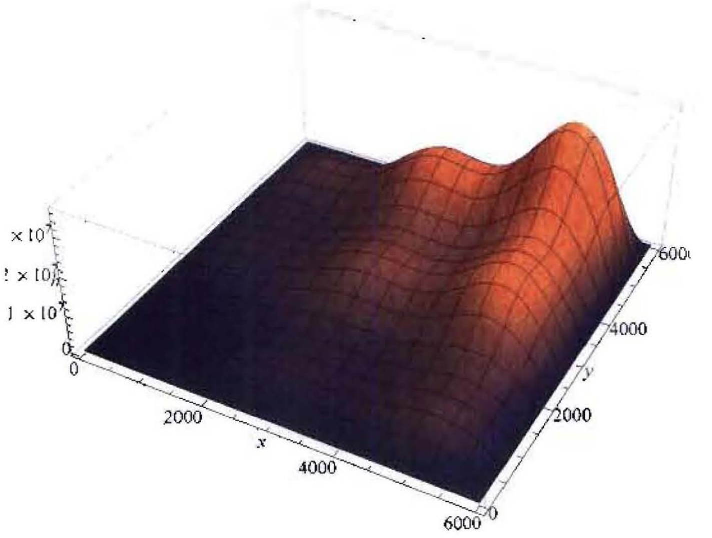
Solving the 2-Dimensional Groundwater Flow Equation

Defining h(x,y,t).

```
h[x_, y_, t_] := w[x, y, t] + ss[x, y]
```

Plotting h(x,y,t) at time t=0.

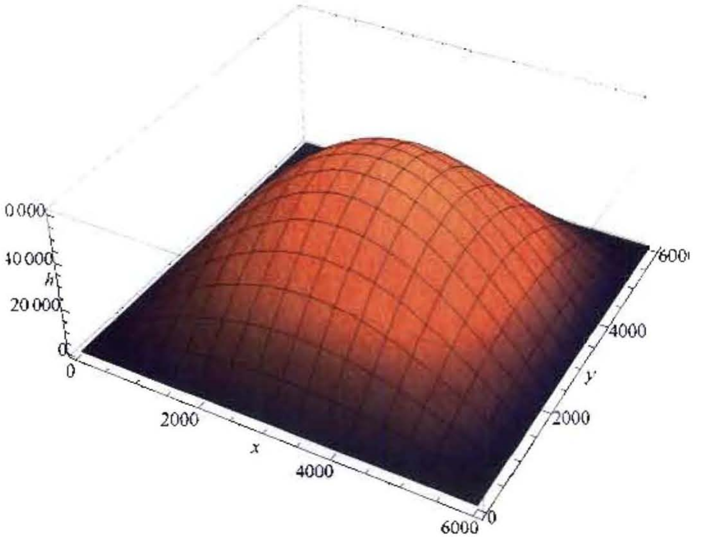
```
h3 = Plot3D[h[x, y, 0], {x, 0, 6000},
  {y, 0, 6000}, AxesLabel -> {x, y, h}, ColorFunction -> "RustTones"]
```



Plotting h(x,y,t) at time t=1000.

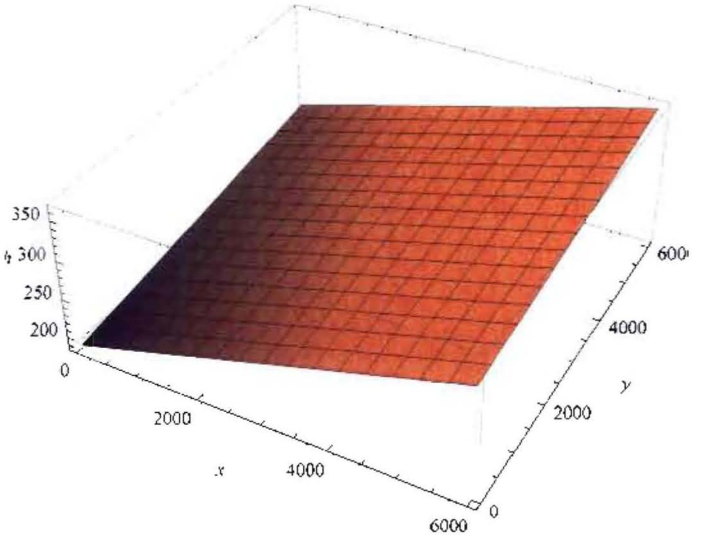
In[55]:=
```
h4 = Plot3D[h[x, y, 1000], {x, 0, 6000},
  {y, 0, 6000}, AxesLabel → {x, y, h}, ColorFunction → "RustTones"]
```

Out[55]=



In[56]:=
```
h4a = Plot3D[h[x, y, 10000], {x, 0, 6000},
  {y, 0, 6000}, AxesLabel → {x, y, h}, ColorFunction → "RustTones"]
```

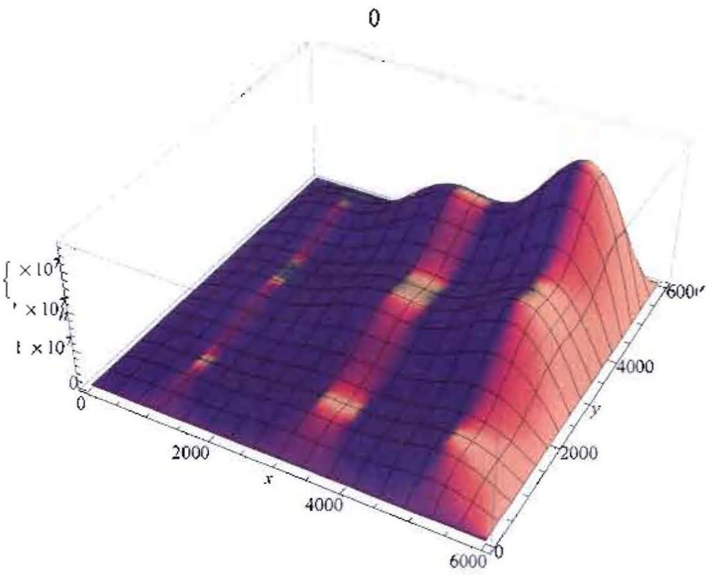
Out[56]=

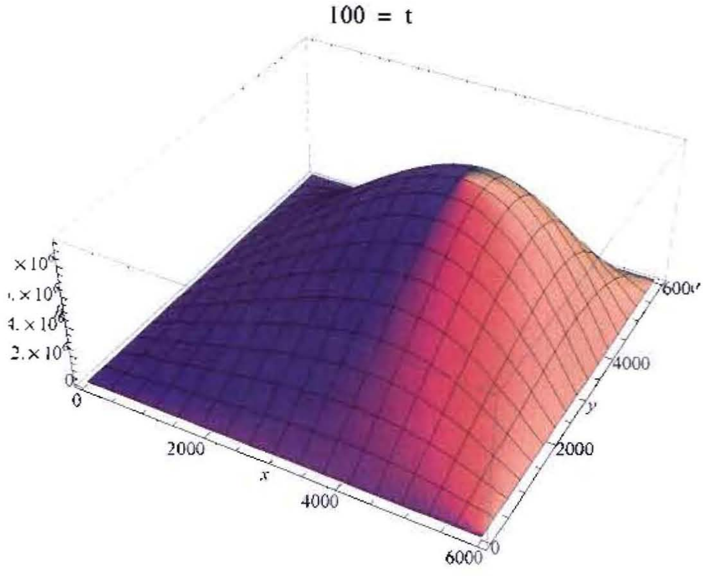


Plotting h(x,y,t) as time increases, to see how it changes.

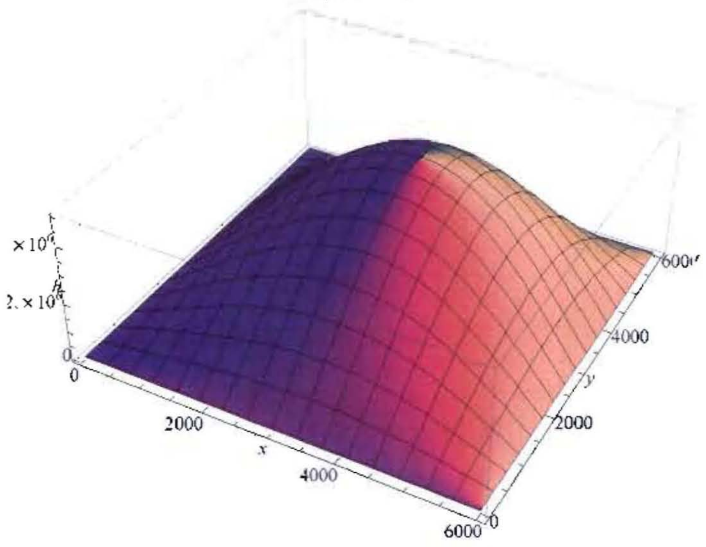
In[57]:=
```
Table[Plot3D[Evaluate[h[x, y, t]], {x, 0, a}, {y, 0, b},
  PlotRange → All, AxesLabel → {x, y, h}, PlotLabel → t " =  t"], {t, 0, 2500, 100}]
```

Out[57]= {

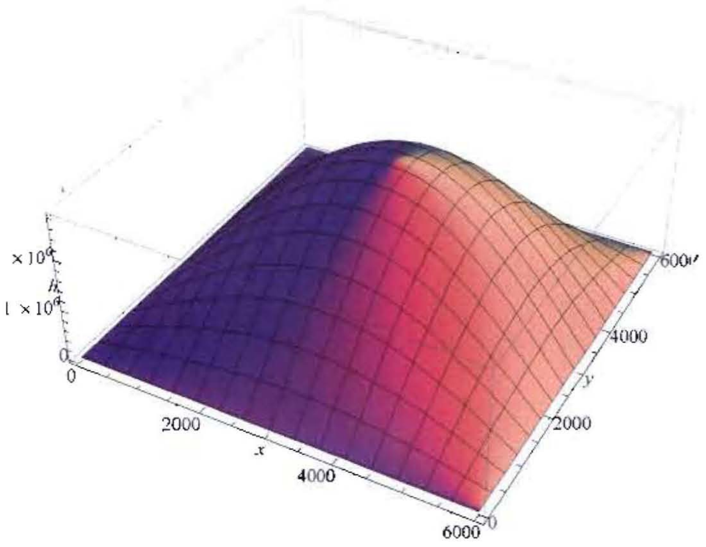

0
2000
x
4000
6000
0
2000
y
4000
0


100 = t
2000
x
4000
6000
0
2000
y
4000
0

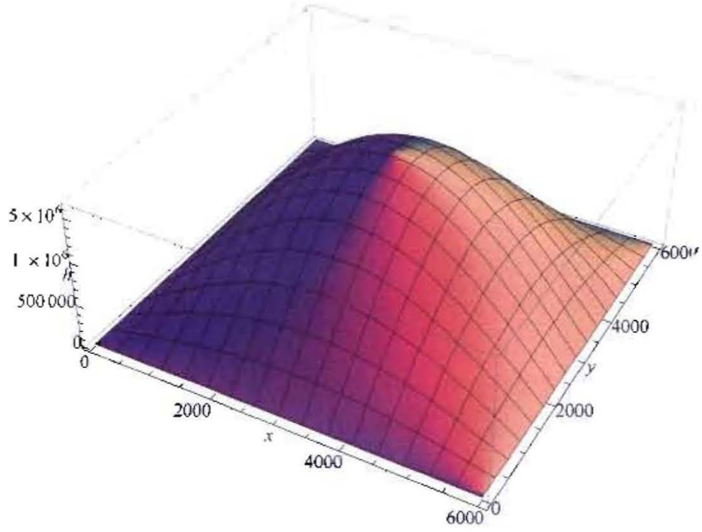
200 = t


0
0
2000
x
4000
6000
0
2000
y
4000
h

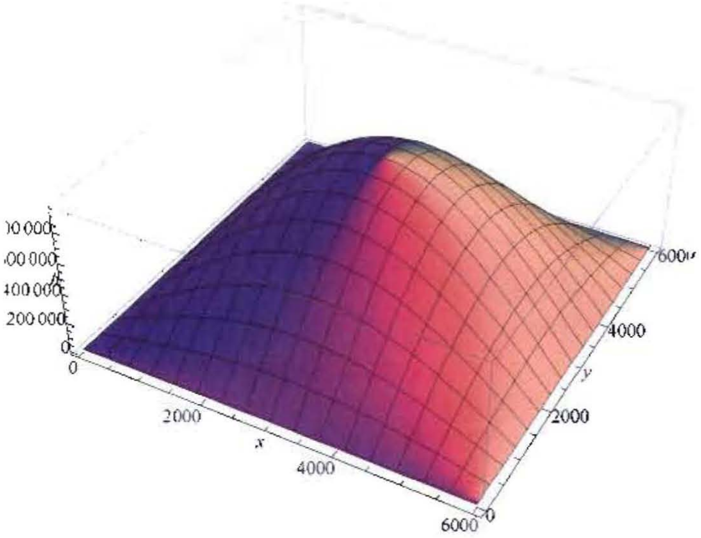
300 = t

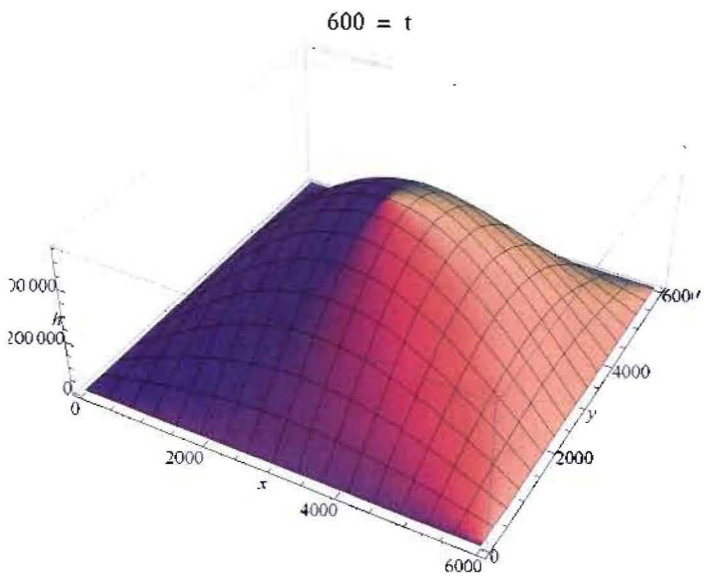

0
0
2000
x
4000
6000
0
2000
y
4000
h

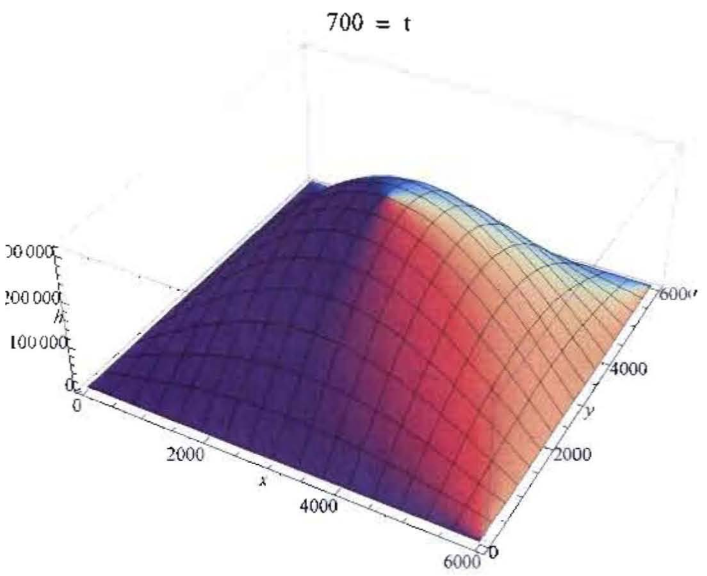
400 = t


500 000
0
2000
x
4000
6000
4000
y
2000
0

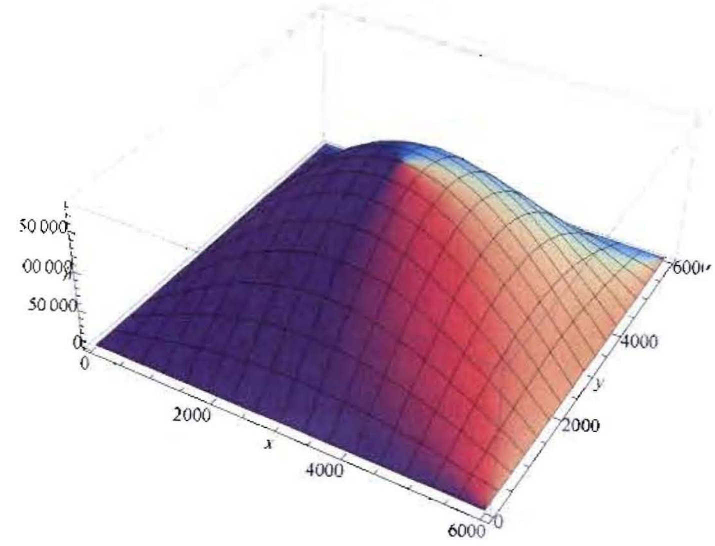
500 = t


200 000
0
2000
x
4000
6000
4000
y
2000
0


600 = t
00 000
h
200 000
0
0
2000
x
4000
6000
6000
4000
y
2000
0

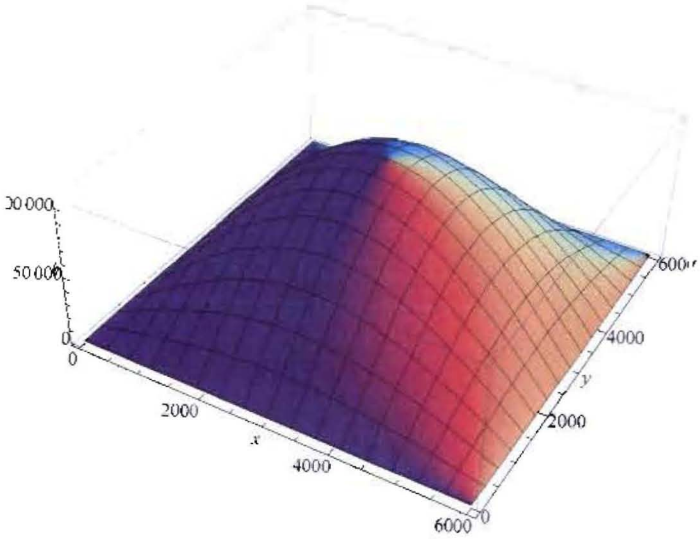

700 = t
00 000
200 000
h
100 000
0
0
2000
x
4000
6000
6000
4000
y
2000
0

800 = t



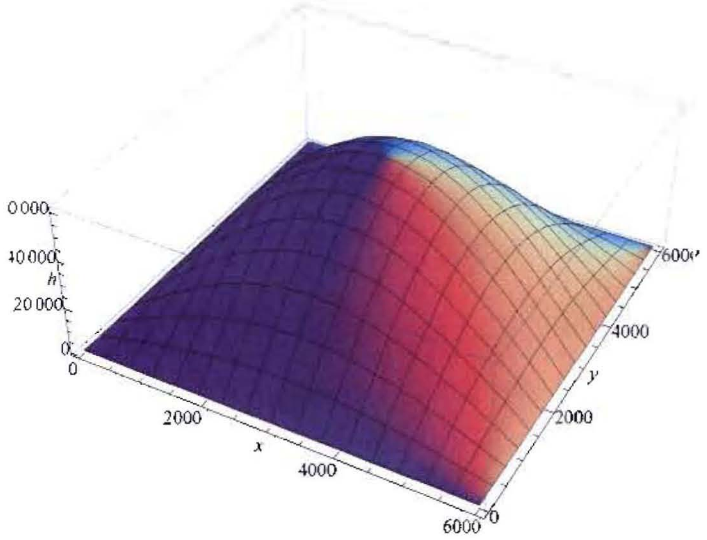
50 000
00 000
50 000
0
0
2000
x
4000
6000
0
2000
y
4000

900 = t

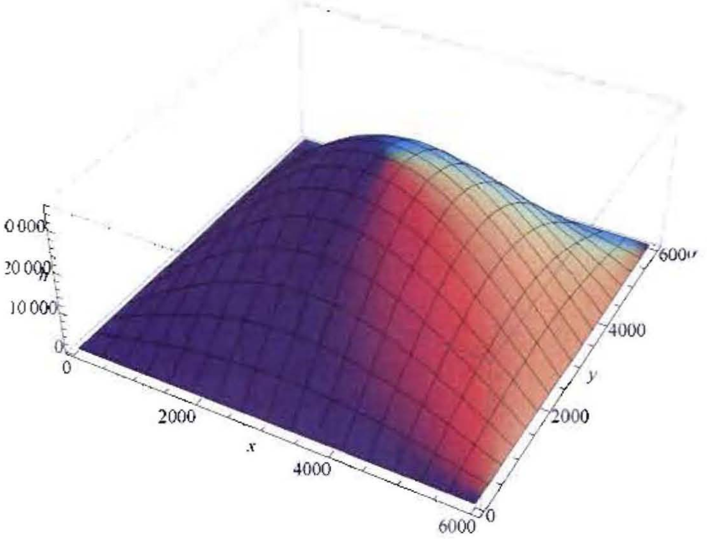


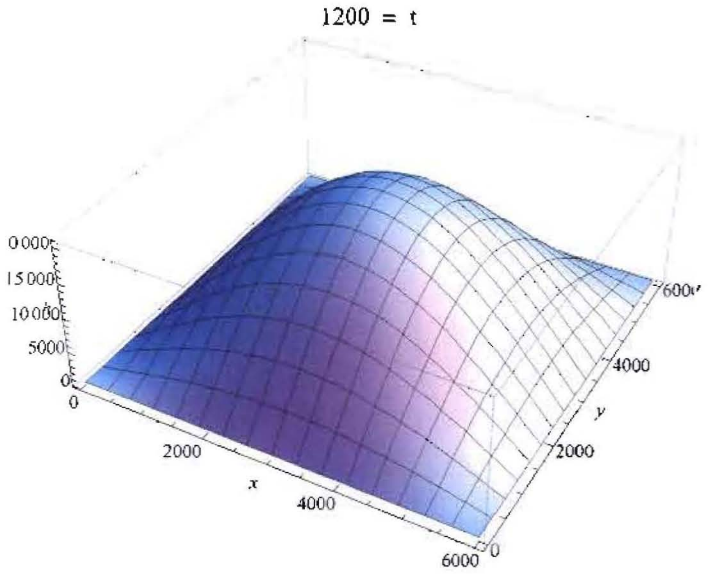
00 000
50 000
0
0
2000
x
4000
6000
0
2000
y
4000

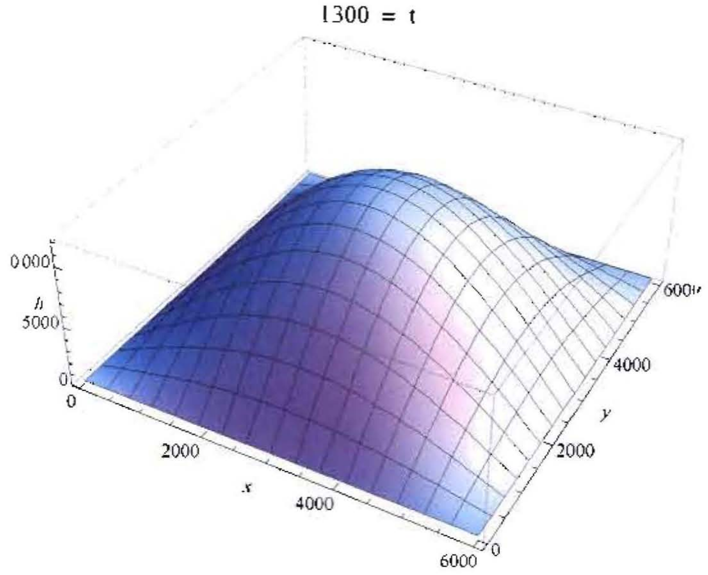
1000 = t

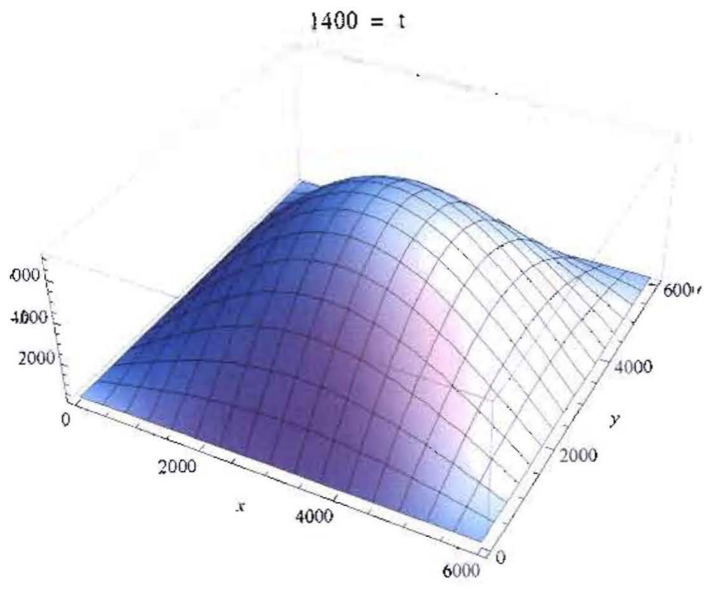

0 000
40 000
h
20 000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

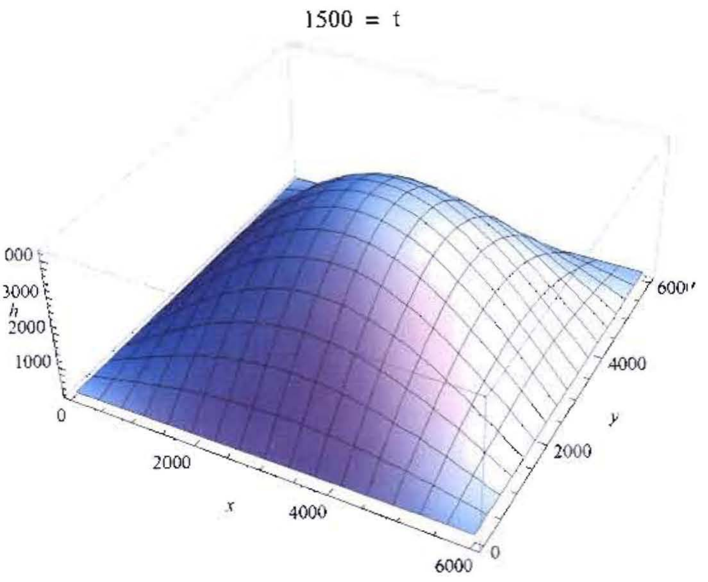
1100 = t


0 000
20 000
h
10 000
0
0
2000
x
4000
6000
0
2000
y
4000
6000

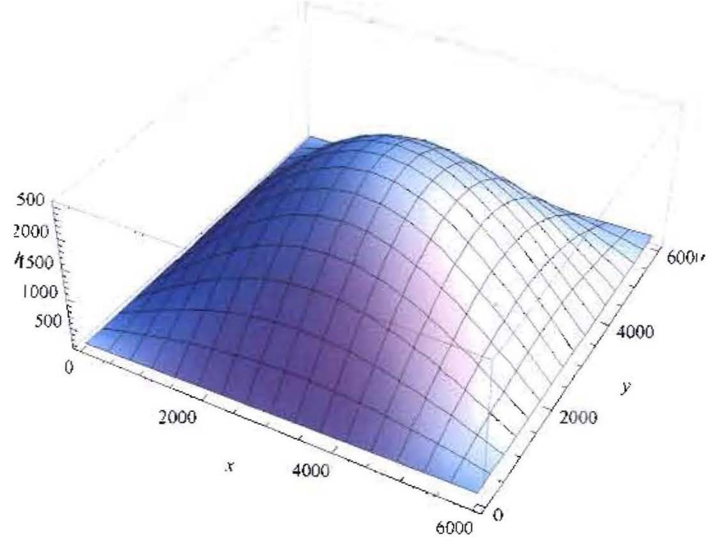

1200 = t
0 000
15 000
10 000
5000
0
0
2000
x
4000
6000
0
2000
y
4000
600


1300 = t
0 000
h
5000
0
0
2000
x
4000
6000
0
2000
y
4000
600

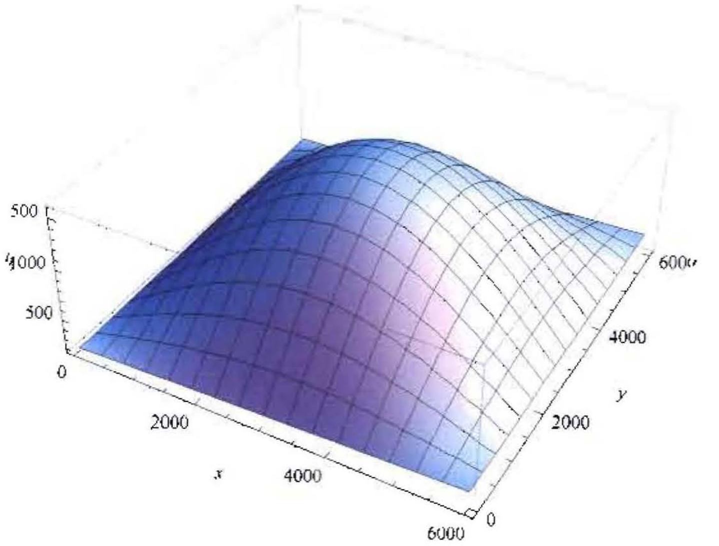

1400 = t
000
4000
2000
0
2000
x
4000
6000
0
2000
y
4000
6000


1500 = t
000
3000
h
2000
1000
0
2000
x
4000
6000
0
2000
y
4000
6000

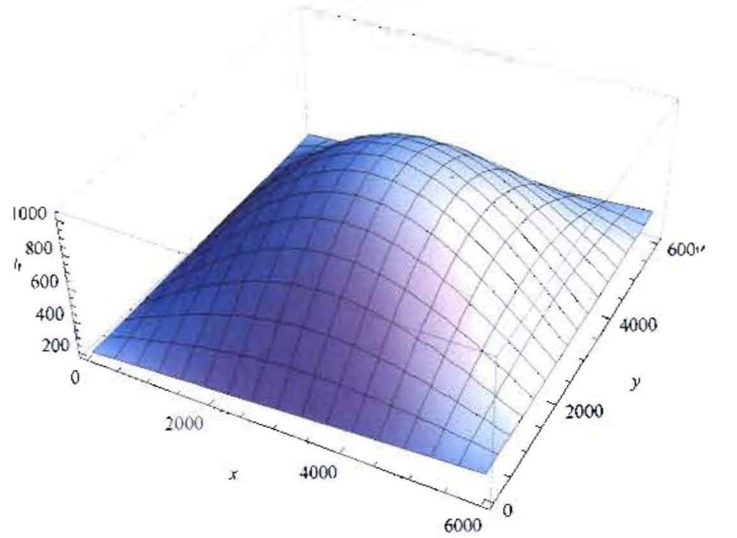
1600 = t


500
2000
1500
1000
500
0
2000
x
4000
6000
0
2000
y
4000

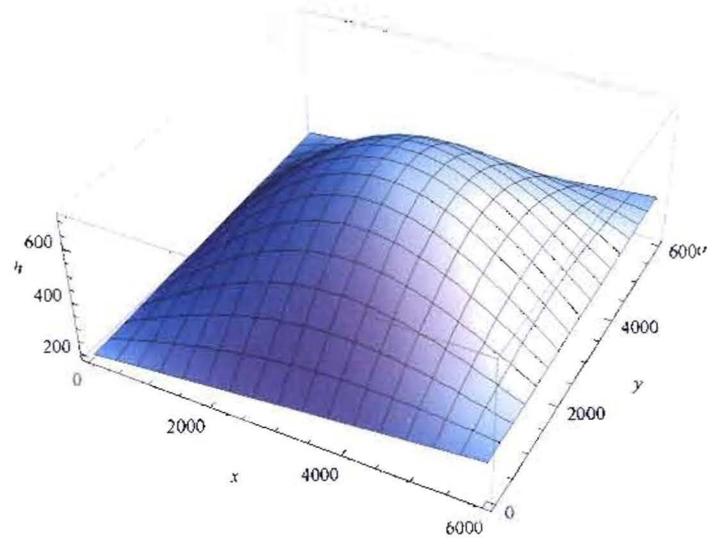
1700 = t

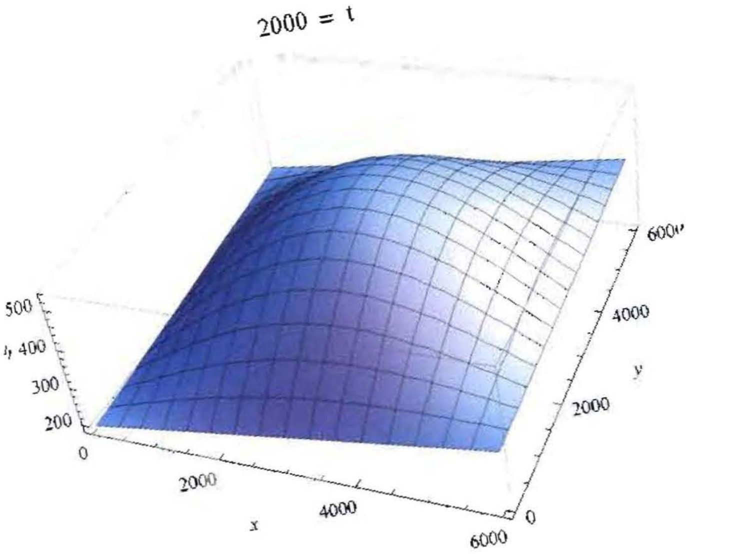

500
500
0
2000
x
4000
6000
0
2000
y
4000

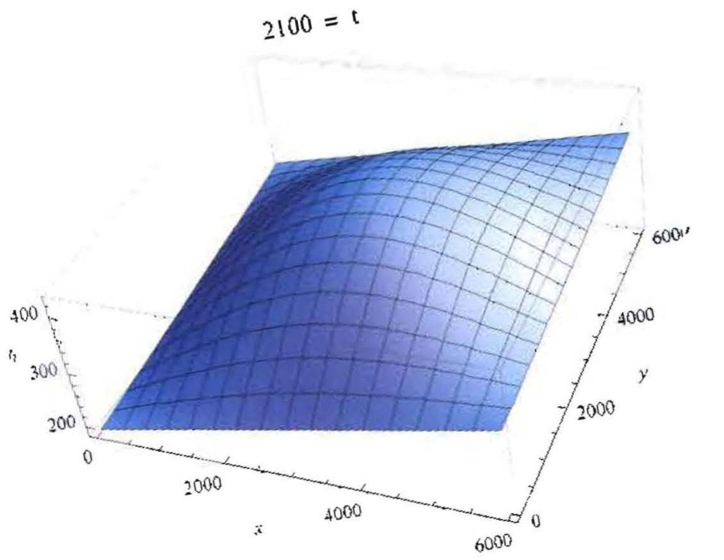
1800 = t

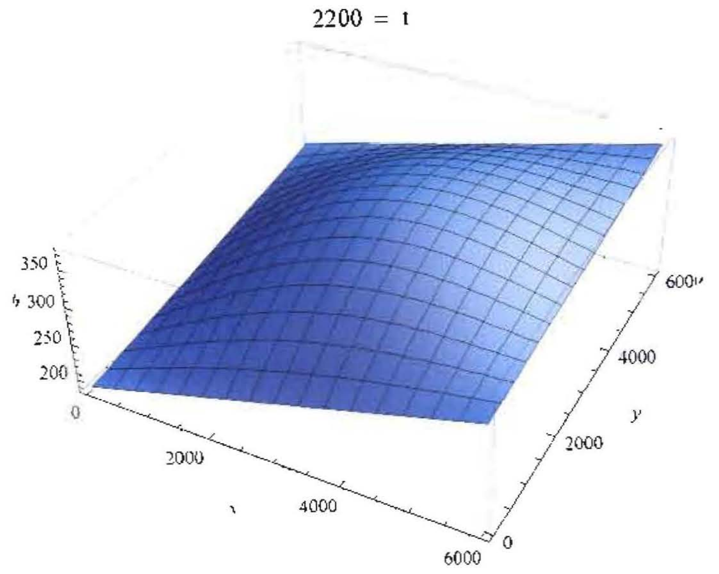

1000
800
h
600
400
200
0
2000
x
4000
6000
0
2000
y
4000
6000

1900 = t

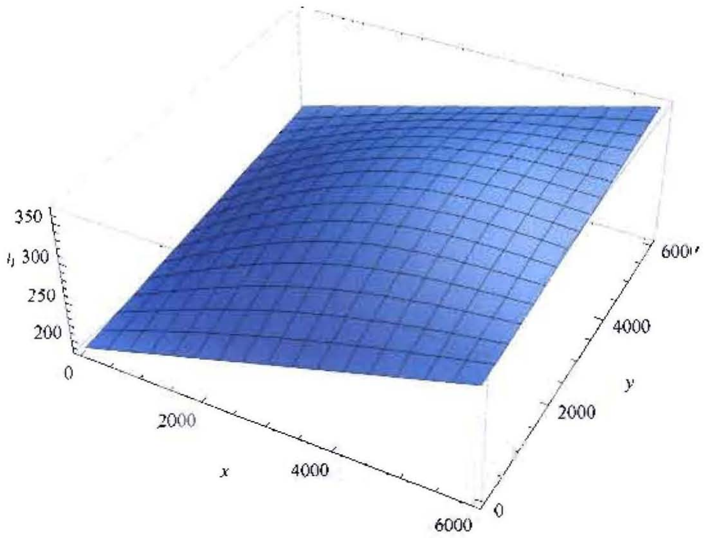

600
h
400
200
0
2000
x
4000
6000
0
2000
y
4000
6000


2000 = t
500
400
300
200
0
2000
4000
6000
x
0
2000
4000
y

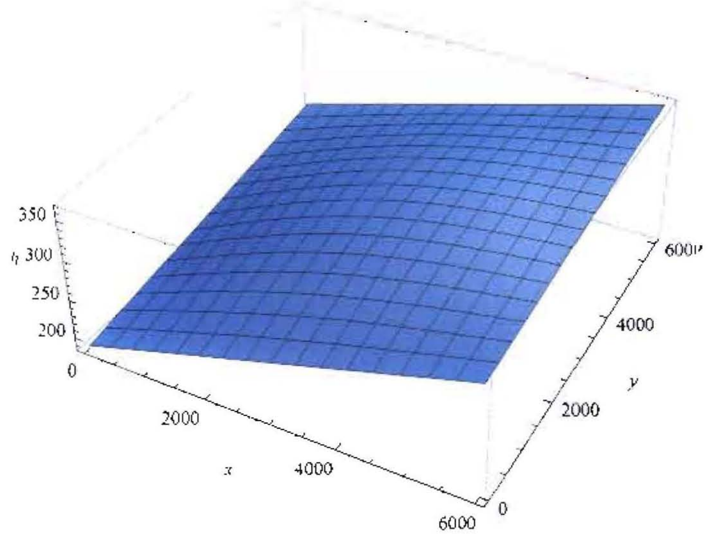

2100 = t
400
300
200
0
2000
4000
6000
x
0
2000
4000
y


2200 = t
350
300
250
200
0
2000
4000
6000
y
2000
4000
0

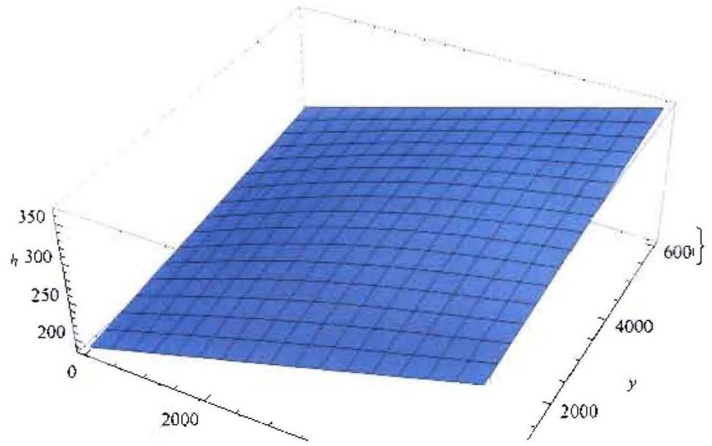
2300 = t



2400 = t



2500 = t



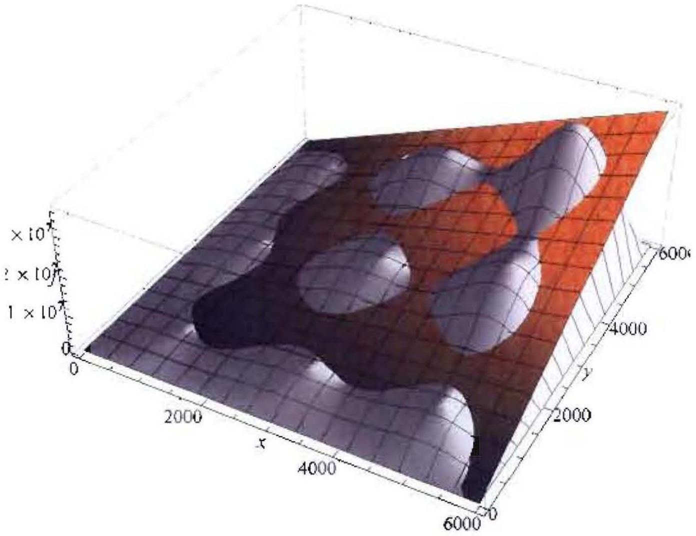
Appendix 4

Comparisons

Comparing h(x,y,0) from fully analytical solution to f(x,y)

In[58]:= `Show[f1, h1]`

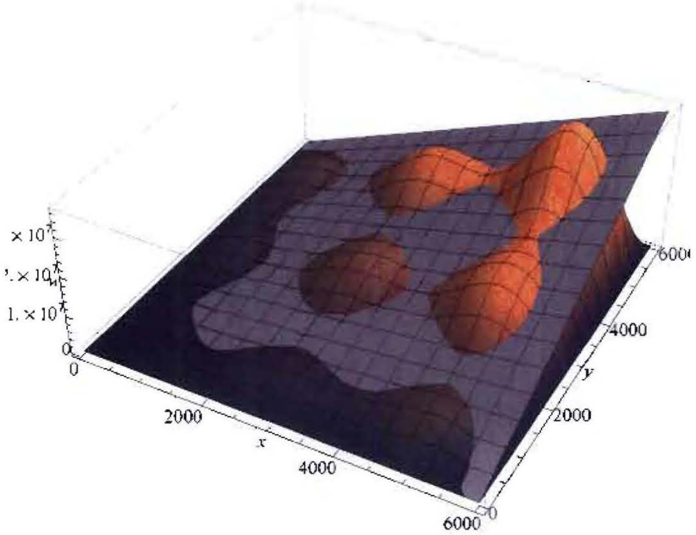
Out[58]=



Comparing w(x,y,0) to g(x,y)

In[59]:= `Show[w1, g1]`

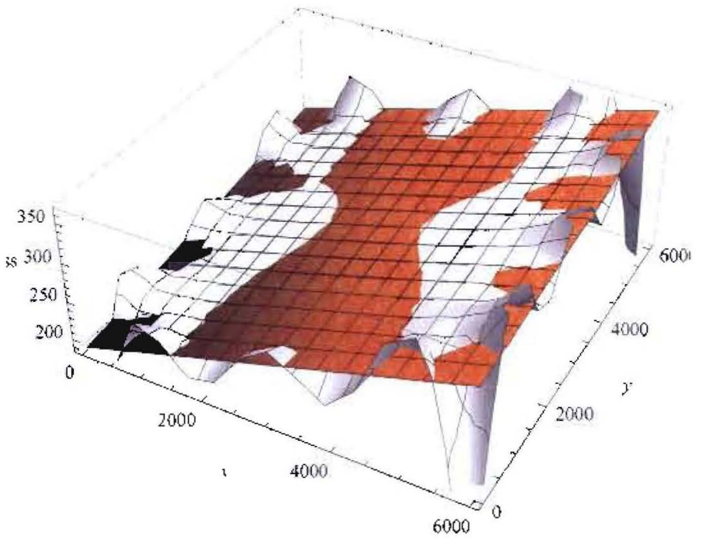
Out[59]=



Comparing ss(x,y) to v(x,y)

In[60]:= `Show[s1, v1]`

Out[60]=

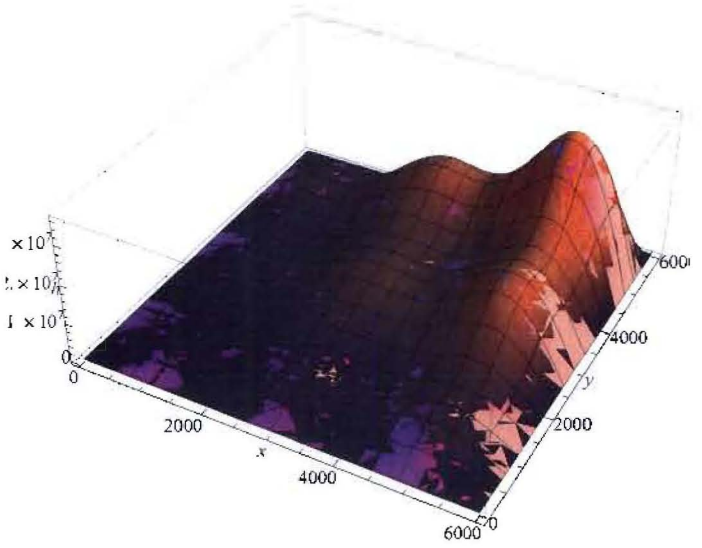


Comparing the results of the two methods

Plotting the graphs of h(x,y,0)

In[61]:= `Show[h1, h3]`

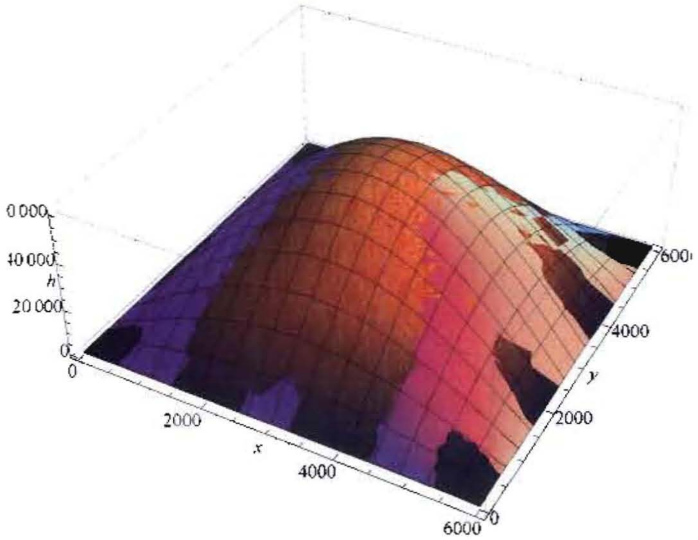
Out[61]=



Plotting the graphs of h(x,y,1000)

In[62]:= `Show[h2, h4]`

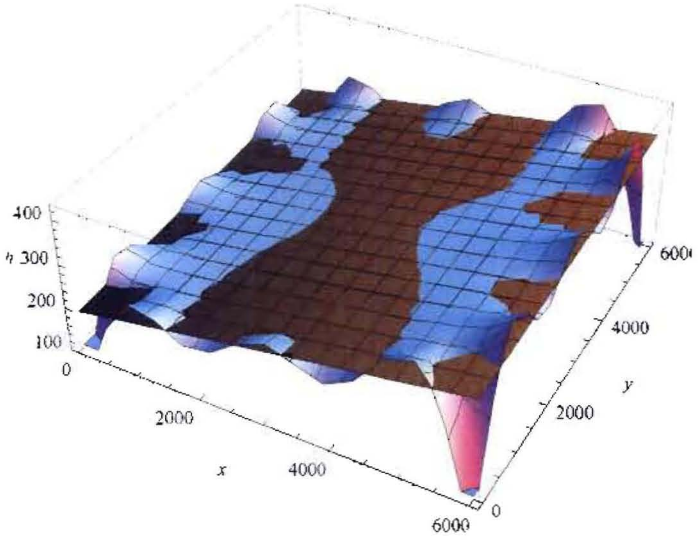
Out[62]=



Plotting the graphs of h(x,y,10000)

In[63]:= `Show[h2a, h4a]`

Out[63]=



Appendix 5

Well	t=0	t=1	t=2	t=3	t=4	t=5	t=6
h1	250	245	246.25	246.71875	246.2695313	247.4169922	247.0361328
h2	240	235	233.75	233.4375	234.9804688	234.5751953	236.8383789
h3	230	225	224.375	225.078125	225.1757813	227.9638672	227.746582
h4	210	215	215.3125	215.9375	218.6328125	218.8378906	221.6113281
h5	210	208.75	210.9375	212.890625	213.2421875	215.1611328	215.2282715
h6	250	260	263.125	261.640625	264.6875	263.5693359	266.1938477
h7	240	245	243.125	248.125	246.8554688	251.9726563	250.7348633
h8	230	227.5	231.25	231.328125	238.2421875	237.5732422	243.4570313
h9	210	217.5	218.4375	226.5625	226.9335938	233.3203125	232.9174805
h10	210	213.75	221.25	222.03125	227.0117188	227.0751953	230.5322266
h11	270	282.5	277.1875	283.90625	281.1523438	285.3857422	283.7207031
h12	260	250	264.375	261.015625	269.9804688	267.2216797	273.6425781
h13	230	237.5	239.375	253.203125	251.328125	260.5712891	258.5498047
h14	220	217.5	238.4375	238.4375	249.3945313	248.1835938	255.4736328
h15	210	233.75	233.75	243.59375	243.125	248.6474609	248.0383301
h16	320	298.75	308.125	301.953125	306.875	304.0917969	307.5012207
h17	260	292.5	284.375	294.6875	289.5507813	296.640625	293.6254883
h18	240	262.5	278.75	274.53125	284.6679688	281.2207031	288.5095215
h19	220	265	262.1875	274.21875	271.3476563	279.3554688	277.331543
h20	260	258.75	269.6875	267.03125	273.1835938	271.8945313	275.9606934
h21	350	342.5	331.25	333.90625	330.6640625	332.9785156	331.5148926
h22	350	326.25	327.5	320.703125	325.0390625	321.9677734	325.5224609
h23	340	320	312.1875	316.5625	312.65625	317.4707031	315.3039551
h24	340	310	310	305.390625	310.1757813	308.0273438	311.9165039
h25	335	310	302.1875	304.921875	303.1054688	305.8398438	304.9804688

Well	t=7	t=8	t=9	t=10	t=11	t=12
h1	248.2580566	247.9380798	248.9933205	248.7286663	249.5774364	249.3656689
h2	236.3793945	238.441925	237.9993248	239.6814823	239.2987823	240.616371
h3	230.4766846	230.0849152	232.3342323	231.9191074	233.6816382	233.3112374
h4	221.4730835	223.757782	223.4733963	225.2546406	224.947871	226.3105077
h5	216.7858887	216.692276	217.8954506	217.7406549	218.6561108	218.4966519
h6	265.3729248	267.5313568	266.9153404	268.6282635	268.1638932	269.4931993
h7	255.032959	253.9743042	257.3983765	256.5473557	259.2064095	258.5425806
h8	242.4871826	247.1372223	246.2037086	249.7904301	248.9982963	251.7326632
h9	237.7685547	237.116394	240.7888794	240.1317215	242.9042816	242.3319483
h10	230.2960205	232.8240204	232.4892235	234.3698025	234.0387368	235.4450664
h11	286.8344116	285.7489777	288.121357	287.3795509	289.1889513	288.6655989
h12	271.6577148	276.4830017	275.0710487	278.725462	277.7093506	280.4726583
h13	265.2706909	263.6392212	268.6402321	267.3950005	271.1383235	270.2003443
h14	254.2092896	259.4364929	258.3605576	262.2022533	261.3428891	264.1935521
h15	251.7416382	251.1482239	253.7948799	253.2825708	255.2198732	254.8035806
h16	305.965271	308.4710693	307.5318146	309.4020796	308.7891519	310.1906741
h17	298.7939453	296.9216919	300.7418823	299.5154953	302.3569489	301.5190649
h18	286.2026978	291.5042114	289.9446869	293.8351488	292.7508414	295.6268966
h19	282.9650879	281.5383911	285.585022	284.5622635	287.5117302	286.7677832
h20	275.0875854	277.9190063	277.2805023	279.3074369	278.8326967	280.3084958
h21	333.2559204	332.4565887	333.745079	333.2615614	334.2167962	333.9040169
h22	323.861084	326.5092468	325.514431	327.4651051	326.8269157	328.2685822
h23	318.9871216	317.6794434	320.3734589	319.5306063	321.5005839	320.9306157
h24	310.6539917	313.4803772	312.6633072	314.7020817	314.1447055	315.6286734
h25	306.9692993	306.4353943	307.8498459	307.4859524	308.5023797	308.2443506

Well	t=13	t=14	t=15	t=16	t=17	t=18
h1	250.0273926	249.8620835	250.3697656	250.2427199	250.6287212	250.5320043
h2	240.3048717	241.3177503	241.0725907	241.8434801	241.6542154	242.2375631
h3	234.6648855	234.3590897	235.3884829	235.145994	235.9245206	235.7365078
h4	226.0349594	227.0685566	226.8403006	227.6207811	227.4394351	228.0272385
h5	219.1888935	219.0482512	219.5700216	219.4545492	219.8470856	219.7557842
h6	269.1434621	270.1613121	269.8982889	270.6714047	270.4738017	271.0581707
h7	260.578723	260.0691897	261.6156718	261.2281477	262.3970108	262.1038772
h8	251.0965277	253.1676246	252.6710847	254.2338211	253.8523806	255.0288121
h9	244.4204473	243.9538616	245.5246198	245.157197	246.337348	246.0536083
h10	235.1580452	236.2115299	235.9778962	236.7675614	236.5837019	237.1757728
h11	290.0391329	289.6618824	290.7001812	290.4243393	291.2069508	291.003375
h12	279.7318972	281.8160003	281.2706264	282.8393375	282.4351109	283.6143122
h13	273.0064426	272.3021977	274.40651	273.8781838	275.456369	275.0600977
h14	263.5259141	265.6507682	265.1395067	266.7272284	266.3389154	267.5269852
h15	256.2367786	255.909472	256.9756042	256.7230614	257.5186574	257.3259508
h16	309.7721702	310.8234124	310.5284417	311.3170609	311.1045873	311.6961702
h17	303.6397028	303.0492359	304.6349871	304.209773	305.3969183	305.0862185
h18	294.854452	296.991647	296.4315175	298.0250891	297.6139837	298.8047999
h19	288.9394045	288.3924955	290.0021794	289.5972193	290.7955665	290.4942988
h20	279.9539286	281.0401188	280.7748429	281.5798398	281.381186	281.9804278
h21	334.6148141	334.4026486	334.9330753	334.784237	335.1808116	335.0739449
h22	327.8384244	328.9088887	328.6085064	329.4061856	329.1911921	329.7870189
h23	322.3810381	321.9821409	323.0566799	322.7707585	323.5703459	323.3620715
h24	315.2356873	316.3261837	316.0430101	316.8501087	316.6431103	317.2433499
h25	308.9842923	308.797404	309.3415756	309.2044632	309.6074871	309.5060741

Well	t=19	t=20	t=21	t=22	t=23	t=24
h1	250.8239335	250.7507279	250.9708039	250.9155885	251.0811715	251.0396148
h2	242.0930973	242.5330129	242.4234895	242.7545455	242.6718556	242.9206691
h3	236.3234035	236.1795238	236.6210911	236.5118419	236.8436676	236.7611057
h4	227.8864751	228.3284646	228.2206705	228.5526928	228.4708101	228.7200742
h5	220.0507528	219.9800777	220.2015694	220.1475354	220.3137782	220.272773
h6	270.9098141	271.3502028	271.2388643	271.5701403	271.4866036	271.7355197
h7	262.9847145	262.7637066	263.4262869	263.2599919	263.7578373	263.6328648
h8	254.7385227	255.6228869	255.4032074	256.0674321	255.9017572	256.4003695
h9	246.9397022	246.7230806	247.3881109	247.2238632	247.7228509	247.5988338
h10	237.0338358	237.4778129	237.3694712	237.7024201	237.6202819	237.8699781
h11	291.5921633	291.4410227	291.8834705	291.7708339	292.1030699	292.0189278
h12	283.3133921	284.1990451	283.9744062	284.6392315	284.471243	284.9701354
h13	276.2437274	275.9465188	276.8342394	276.611332	277.2771221	277.1099415
h14	267.2334889	268.1232792	267.9021035	268.5688588	268.402486	268.9022787
h15	257.9207964	257.7747264	258.22	258.1097288	258.4432831	258.3602444
h16	311.5408846	311.984634	311.8700651	312.2029077	312.1178645	312.3675112
h17	305.9755753	305.7463768	306.4129292	306.2428142	306.7425121	306.6157578
h18	298.5006716	299.3917464	299.165611	299.8329662	299.6642796	300.1643523
h19	291.3888907	291.1640881	291.8330851	291.6650198	292.1658587	292.0400604
h20	281.8315809	282.2789079	282.1673406	282.5018534	282.41821	282.668636
h21	335.3707973	335.2928608	335.5152313	335.4578099	335.6244627	335.5818772
h22	329.6305587	330.0762912	329.9611745	330.2949429	330.2096441	330.4597228
h23	323.9587922	323.8054604	324.2516111	324.1379524	324.4719163	324.3872974
h24	317.0906111	317.5384068	317.4250241	317.7597562	317.6752659	317.9257943
h25	309.8059444	309.730548	309.9543287	309.8980912	310.0654024	310.023369

Well	t=25	t=26	t=27	t=28	t=29	t=30
h1	251.1640472	251.132812	251.2262507	251.2027928	251.2729251	251.2553169
h2	242.8583963	243.0452497	242.998426	243.1386795	243.1035062	243.2087492
h3	237.0102782	236.9480652	237.135086	237.0882902	237.2286218	237.1934615
h4	228.6581781	228.8452416	228.7985937	228.9389452	228.9038539	229.0091426
h5	220.3975131	220.3665352	220.4601174	220.4367795	220.5069787	220.4894266
h6	271.6728518	271.859753	271.8127451	271.9530208	271.9177616	272.023015
h7	264.0066734	263.9128269	264.1933813	264.1229418	264.33345	264.2805949
h8	256.2756865	256.6498527	256.5561412	256.8368625	256.766486	256.977072
h9	247.9731751	247.8797744	248.1605773	248.0903457	248.3009698	248.2482117
h10	237.8079628	237.9952278	237.9485244	238.0889698	238.0538526	238.1591852
h11	292.2682915	292.2053414	292.3924514	292.3453117	292.4856849	292.4503642
h12	284.844373	285.2186698	285.1244549	285.405237	285.3346257	285.5452401
h13	277.609284	277.4838984	277.8584053	277.7643661	278.0452463	277.9747169
h14	268.77727	269.1519869	269.0581235	269.3391017	269.2686543	269.4793601
h15	258.6102232	258.5477878	258.7351847	258.6882852	258.8287922	258.7935835
h16	312.3041407	312.4913826	312.4440469	312.5844816	312.5490694	312.654397
h17	306.9904304	306.8957527	307.1767102	307.105883	307.3165791	307.2635432
h18	300.0382643	300.4131118	300.3187449	300.5997839	300.5291017	300.7398359
h19	292.4152654	292.3210336	292.6022394	292.5316202	292.7424323	292.6894933
h20	282.6059184	282.793524	282.7464929	282.8870972	282.8518271	282.9572338
h21	335.7068085	335.6750934	335.7687648	335.745083	335.8153239	335.7976113
h22	330.3962331	330.5836766	330.5362852	330.676814	330.6413759	330.7467473
h23	324.6374674	324.5742948	324.7617809	324.7145374	324.855086	324.8197169
h24	317.8626817	318.0503351	318.0031196	318.1437463	318.1083902	318.2138073
h25	310.1486076	310.11715	310.2109648	310.1874031	310.2577109	310.2400543

Well	t=31	t=32	t=33	t=34	t=35	t=36
h1	251.307941	251.2947281	251.3342078	251.3242948	251.35391	251.3464738
h2	243.1823433	243.2613003	243.2414838	243.300713	243.285845	243.3302722
h3	237.2987409	237.2723411	237.3513151	237.3315014	237.3907385	237.3758718
h4	228.982775	229.0617532	229.0419546	229.1011937	229.086334	229.130766
h5	220.5420819	220.5288951	220.5683893	220.5584886	220.5881106	220.5806801
h6	271.996569	272.0755308	272.0556956	272.114927	272.1000503	272.1444786
h7	264.438519	264.3988658	264.5173291	264.4875836	264.5764404	264.5541287
h8	256.9242462	257.0822067	257.0425672	257.1610474	257.1313084	257.2201731
h9	248.40619	248.366582	248.4850705	248.4553461	248.5442147	248.5219129
h10	238.1328055	238.2118042	238.1919999	238.2512486	238.2363863	238.2808226
h11	292.555663	292.5291884	292.6081713	292.5883228	292.6475641	292.6326811
h12	285.4923048	285.6502785	285.6105879	285.7290743	285.6993114	285.788179
h13	278.185377	278.13248	278.2904751	278.2508023	278.3692986	278.339544
h14	269.4265014	269.5845178	269.5448628	269.6633691	269.6336228	269.7224997
h15	258.8989448	258.8725224	258.9515345	258.9317103	258.9909653	258.9760937
h16	312.6278797	312.706876	312.6870075	312.7462552	312.7313629	312.7757988
h17	307.4215551	307.3818175	307.5003216	307.4705368	307.5594127	307.5370827
h18	300.6868676	300.8448973	300.8051912	300.9237037	300.8939335	300.9828133
h19	292.8475593	292.807867	292.9263964	292.8966326	292.9855203	292.9632001
h20	282.9307828	283.0098161	282.9899785	283.0492434	283.0343655	283.0788094
h21	335.8502861	335.8370244	335.8765277	335.866592	335.8962182	335.8887714
h22	330.7202179	330.7992347	330.7793606	330.8386177	330.8237228	330.8681631
h23	324.9250976	324.8986004	324.9776216	324.9577624	325.0170216	325.0021337
h24	318.1873161	318.2663543	318.246498	318.3057651	318.2908786	318.3353235
h25	310.2927603	310.2795247	310.3190426	310.3091191	310.3387521	310.331311

Well	t=37	t=38	t=39	t=40	t=41	t=42
h1	251.3686877	251.3631099	251.3797715	251.3755878	251.3880845	251.3849466
h2	243.3191186	243.3524416	243.3440751	243.3690685	243.3627931	243.3815387
h3	237.4203028	237.4091498	237.4424745	237.4341083	237.4591025	237.4528272
h4	229.1196162	229.1529413	229.1445767	229.1695711	229.1632965	229.1820426
h5	220.6028971	220.597322	220.613985	220.6098026	220.6223	220.6191627
h6	272.1333209	272.1666444	272.158276	272.1832697	272.1769933	272.195739
h7	264.6207757	264.6040408	264.654028	264.6414762	264.6789677	264.6695536
h8	257.1978644	257.264515	257.2477815	257.2977705	257.2852193	257.3227115
h9	248.5885653	248.571835	248.6218248	248.6092752	248.6467678	248.6373547
h10	238.2696717	238.3029989	238.2946336	238.319629	238.3133542	238.3321007
h11	292.6771141	292.6659535	292.6992791	292.6909094	292.715904	292.7096271
h12	285.7658591	285.8325112	285.8157724	285.865762	285.8532085	285.890701
h13	278.4284163	278.4061003	278.4727545	278.4560176	278.5060082	278.4934555
h14	269.7001877	269.766844	269.7501089	269.8001005	269.7875486	269.8250421
h15	259.0205329	259.0093776	259.0427062	259.034339	259.059335	259.0530592
h16	312.7646338	312.7979608	312.789589	312.8145843	312.8083064	312.8270529
h17	307.6037385	307.586995	307.6369864	307.6244307	307.661924	307.652508
h18	300.9604901	301.0271478	301.0104075	301.0603998	301.0478455	301.0853392
h19	293.0298615	293.0131226	293.0631165	293.0505629	293.0880574	293.0786425
h20	283.0676512	283.1009819	283.0926133	283.1176103	283.1113339	283.1300812
h21	335.9109905	335.9054077	335.9220717	335.9178857	335.9303836	335.9272446
h22	330.856997	330.890326	330.8819537	330.9069499	330.9006718	330.9194187
h23	325.046575	325.0354121	325.0687416	325.0603708	325.0853672	325.0790898
h24	318.3241612	318.3574924	318.3491219	318.3741192	318.3678419	318.3865893
h25	310.3535332	310.3479531	310.3646186	310.3604338	310.3729324	310.3697939

Well	t=43	t=44	t=45	t=46	t=47	t=48
h1	251.3943194	251.3919659	251.3989956	251.3972305	251.4025028	251.4011789
h2	243.3768318	243.3908913	243.387361	243.3979058	243.395258	243.4031666
h3	237.4715732	237.4668664	237.4809261	237.4773958	237.4879406	237.4852929
h4	229.1773361	229.1913958	229.1878657	229.1984106	229.1957629	229.2036715
h5	220.6285358	220.6261826	220.6332125	220.6314474	220.6367198	220.635396
h6	272.1910318	272.2050913	272.2015609	272.2121056	272.2094578	272.2173664
h7	264.6976726	264.6906119	264.7117013	264.7064057	264.7222229	264.7182512
h8	257.3132977	257.3414171	257.3343565	257.3554462	257.3501507	257.3659679
h9	248.6654742	248.658414	248.6795037	248.6742083	248.6900257	248.686054
h10	238.3273941	238.341454	238.3379239	238.3484688	238.3458211	238.3537298
h11	292.7283732	292.7236657	292.7377254	292.7341948	292.7447397	292.7420918
h12	285.881286	285.9094055	285.9023445	285.9234342	285.9181384	285.9339557
h13	278.5309484	278.5215339	278.5496536	278.5425927	278.5636825	278.5583869
h14	269.815628	269.8437479	269.8366872	269.8577771	269.8524815	269.8682989
h15	259.071806	259.067099	259.081159	259.0776287	259.0881737	259.0855259
h16	312.8223449	312.8364048	312.832874	312.8434189	312.8407709	312.8486796
h17	307.6806279	307.6735664	307.6946562	307.6893603	307.7051776	307.7012057
h18	301.0759239	301.104044	301.0969828	301.1180727	301.1127769	301.1285943
h19	293.1067629	293.0997018	293.120792	293.1154962	293.1313137	293.1273419
h20	283.1253739	283.1394341	283.1359037	283.1464487	283.1438009	283.1517097
h21	335.9366179	335.9342639	335.9412939	335.9395284	335.9448009	335.9434769
h22	330.9147106	330.9287707	330.9252398	330.9357848	330.9331368	330.9410455
h23	325.0978368	325.093129	325.1071891	325.1036584	325.1142035	325.1115555
h24	318.3818816	318.3959418	318.3924112	318.4029563	318.4003083	318.4082171
h25	310.3791676	310.3768139	310.383844	310.3820787	310.3873513	310.3860273

Well	t=49	t=50	t=51	t=52	t=53	t=54
h1	251.4051332	251.4041403	251.407106	251.4063613	251.4085856	251.4080271
h2	243.4011808	243.4071122	243.4056228	243.4100714	243.4089544	243.4122909
h3	237.4932015	237.4912157	237.4971472	237.4956578	237.5001064	237.4989894
h4	229.2016857	229.2076172	229.2061278	229.2105765	229.2094594	229.2127959
h5	220.6393503	220.6383574	220.6413232	220.6405785	220.6428028	220.6422443
h6	272.2153805	272.2213119	272.2198225	272.2242711	272.2231541	272.2264906
h7	264.7301141	264.7271353	264.7360326	264.7337985	264.7404714	264.7387958
h8	257.3619963	257.3738592	257.3708804	257.3797777	257.3775436	257.3842165
h9	248.697917	248.6949383	248.7038355	248.7016015	248.7082744	248.7065988
h10	238.351744	238.3576755	238.3561861	238.3606347	238.3595177	238.3628542
h11	292.7500004	292.7480145	292.753946	292.7524566	292.7569052	292.7557881
h12	285.9299839	285.9418469	285.938868	285.9477653	285.9455312	285.9522041
h13	278.5742042	278.5702325	278.5820955	278.5791166	278.5880139	278.5857798
h14	269.8643272	269.8761902	269.8732114	269.8821087	269.8798746	269.8865475
h15	259.0934346	259.0914487	259.0973803	259.0958909	259.1003395	259.0992224
h16	312.8466936	312.8526251	312.8511357	312.8555843	312.8544672	312.8578037
h17	307.7130688	307.7100899	307.7189872	307.716753	307.723426	307.7217504
h18	301.1246225	301.1364856	301.1335067	301.142404	301.1401698	301.1468428
h19	293.139205	293.1362262	293.1451235	293.1428893	293.1495623	293.1478867
h20	283.1497238	283.1556553	283.1541659	283.1586146	283.1574975	283.160834
h21	335.9474313	335.9464383	335.949404	335.9486593	335.9508836	335.9503251
h22	330.9390595	330.9449911	330.9435016	330.9479503	330.9468332	330.9501697
h23	325.1194642	325.1174783	325.1234098	325.1219204	325.126369	325.125252
h24	318.4062312	318.4121627	318.4106733	318.415122	318.4140049	318.4173414
h25	310.3899817	310.3889887	310.3919545	310.3912098	310.3934341	310.3928756

Well	t=55	t=56	t=57	t=58	t=59	t=60
h1	251.4096954	251.4092765	251.4105276	251.4102135	251.4111518	251.4109162
h2	243.4114531	243.4139554	243.4133271	243.4152038	243.4147326	243.4161402
h3	237.5023258	237.501488	237.5039904	237.503362	237.5052388	237.5047675
h4	229.2119581	229.2144605	229.2138321	229.2157089	229.2152376	229.2166452
h5	220.6439125	220.6434936	220.6447448	220.6444306	220.645369	220.6451334
h6	272.2256528	272.2281551	272.2275268	272.2294035	272.2289323	272.2303398
h7	264.7438005	264.7425438	264.7462974	264.7453548	264.74817	264.7474631
h8	257.3825409	257.3875457	257.386289	257.3900425	257.3891	257.3919151
h9	248.7116035	248.7103468	248.7141004	248.7131579	248.715973	248.7152661
h10	238.3620164	238.3645187	238.3638904	238.3657672	238.3652959	238.3667035
h11	292.7591246	292.7582868	292.7607891	292.7601608	292.7620376	292.7615663
h12	285.9505285	285.9555332	285.9542765	285.9580301	285.9570875	285.9599027
h13	278.5924527	278.5907772	278.5957819	278.5945252	278.5982787	278.5973362
h14	269.8848719	269.8898767	269.88862	269.8923735	269.891431	269.8942461
h15	259.1025589	259.1017211	259.1042235	259.1035951	259.1054719	259.1050006
h16	312.8569659	312.8594682	312.8588399	312.8607167	312.8602454	312.861653
h17	307.7267551	307.7254984	307.7292519	307.7283094	307.7311245	307.7304176
h18	301.1451672	301.1501719	301.1489152	301.1526688	301.1517262	301.1545414
h19	293.1528914	293.1516347	293.1553883	293.1544457	293.1572609	293.156554
h20	283.1599962	283.1624985	283.1618702	283.163747	283.1632757	283.1646833
h21	335.9519933	335.9515744	335.9528256	335.9525114	335.9534498	335.9532142
h22	330.9493319	330.9518342	330.9512059	330.9530826	330.9526114	330.9540189
h23	325.1285885	325.1277507	325.130253	325.1296247	325.1315014	325.1310302
h24	318.4165036	318.4190059	318.4183776	318.4202543	318.4197831	318.4211907
h25	310.3945438	310.3941249	310.3953761	310.3950619	310.3960003	310.3957647

Well	t=61	t=62	t=63	t=64	t=65	t=66
h1	251.41162	251.4114433	251.4119711	251.4118386	251.4122345	251.4121351
h2	243.4157867	243.4168424	243.4165773	243.4173691	243.4171703	243.4177641
h3	237.5061751	237.5058217	237.5068774	237.5066123	237.507404	237.5072052
h4	229.2162918	229.2173474	229.2170824	229.2178741	229.2176753	229.2182691
h5	220.6458372	220.6456605	220.6461883	220.6460558	220.6464516	220.6463522
h6	272.2299864	272.2310421	272.230777	272.2315688	272.23137	272.2319638
h7	264.7495745	264.7490443	264.7506278	264.7502302	264.7514178	264.7511196
h8	257.3912082	257.3933196	257.3927894	257.394373	257.3939753	257.395163
h9	248.7173775	248.7168473	248.7184308	248.7180332	248.7192209	248.7189226
h10	238.36635	238.3674057	238.3671406	238.3679324	238.3677336	238.3683274
h11	292.7629739	292.7626204	292.7636761	292.763411	292.7642028	292.764004
h12	285.9591958	285.9613072	285.960777	285.9623605	285.9619629	285.9631505
h13	278.6001513	278.5994444	278.6015558	278.6010256	278.6026092	278.6022115
h14	269.8935392	269.8956506	269.8951204	269.896704	269.8963063	269.897494
h15	259.1064082	259.1060548	259.1071105	259.1068454	259.1076371	259.1074383
h16	312.8612995	312.8623552	312.8620901	312.8628819	312.8626831	312.8632769
h17	307.732529	307.7319988	307.7335824	307.7331847	307.7343724	307.7340742
h18	301.1538345	301.1559459	301.1554157	301.1569992	301.1566016	301.1577892
h19	293.1586654	293.1581352	293.1597187	293.1593211	293.1605087	293.1602105
h20	283.1643298	283.1653855	283.1651204	283.1659122	283.1657134	283.1663072
h21	335.953918	335.9537413	335.9542691	335.9541366	335.9545324	335.954433
h22	330.9536655	330.9547212	330.9544561	330.9552479	330.955049	330.9556429
h23	325.1324378	325.1320843	325.13314	325.1328749	325.1336667	325.1334679
h24	318.4208372	318.4218929	318.4216278	318.4224196	318.4222208	318.4228146
h25	310.3964685	310.3962918	310.3968196	310.3966871	310.3970829	310.3969835

Well	t=67	t=68	t=69	t=70	t=71	t=72
h1	251.412432	251.4123574	251.4125801	251.4125242	251.4126912	251.4126493
h2	243.417615	243.4180603	243.4179485	243.4182825	243.4181987	243.4184492
h3	237.507799	237.5076499	237.5080953	237.5079835	237.5083175	237.5082336
h4	229.21812	229.2185654	229.2184536	229.2187876	229.2187037	229.2189542
h5	220.6466491	220.6465746	220.6467973	220.6467413	220.6469084	220.6468664
h6	272.2318147	272.23226	272.2321482	272.2324822	272.2323983	272.2326489
h7	264.7520103	264.7517867	264.7524547	264.752287	264.752788	264.7526622
h8	257.3948648	257.3957555	257.3955318	257.3961999	257.3960321	257.3965332
h9	248.7198134	248.7195897	248.7202578	248.72009	248.720591	248.7204652
h10	238.3681783	238.3686237	238.3685118	238.3688459	238.368762	238.3690125
h11	292.7645978	292.7644487	292.7648941	292.7647822	292.7651163	292.7650324
h12	285.9628523	285.9637431	285.9635194	285.9641874	285.9640197	285.9645207
h13	278.6033992	278.603101	278.6039917	278.603768	278.6044361	278.6042683
h14	269.8971958	269.8980865	269.8978628	269.8985309	269.8983631	269.8988642
h15	259.1080321	259.107883	259.1083284	259.1082166	259.1085506	259.1084667
h16	312.8631278	312.8635732	312.8634613	312.8637954	312.8637115	312.863962
h17	307.7349649	307.7347412	307.7354093	307.7352415	307.7357426	307.7356167
h18	301.157491	301.1583817	301.1581581	301.1588261	301.1586584	301.1591594
h19	293.1611013	293.1608776	293.1615456	293.1613779	293.1618789	293.1617531
h20	283.1661581	283.1666035	283.1664916	283.1668257	283.1667418	283.1669923
h21	335.9547299	335.9546554	335.9548781	335.9548222	335.9549892	335.9549472
h22	330.9554938	330.9559391	330.9558273	330.9561613	330.9560774	330.956328
h23	325.1340617	325.1339126	325.1343579	325.1342461	325.1345801	325.1344962
h24	318.4226655	318.4231108	318.422999	318.423333	318.4232492	318.4234997
h25	310.3972804	310.3972059	310.3974286	310.3973727	310.3975397	310.3974977

Well	t=73	t=74	t=75	t=76	t=77	t=78
h1	251.4127745	251.4127431	251.412837	251.4128134	251.4128839	251.4128662
h2	243.4183863	243.4185742	243.418527	243.4186679	243.4186325	243.4187382
h3	237.5084841	237.5084212	237.5086091	237.5085619	237.5087029	237.5086675
h4	229.2188913	229.2190792	229.219032	229.2191729	229.2191376	229.2192432
h5	220.6469917	220.6469602	220.6470542	220.6470306	220.647101	220.6470833
h6	272.232586	272.2327738	272.2327267	272.2328676	272.2328322	272.2329379
h7	264.753038	264.7529436	264.7532255	264.7531547	264.7533661	264.753313
h8	257.3964073	257.3967831	257.3966888	257.3969706	257.3968998	257.3971112
h9	248.720841	248.7207467	248.7210285	248.7209577	248.7211691	248.721116
h10	238.3689496	238.3691375	238.3690903	238.3692312	238.3691958	238.3693015
h11	292.7652829	292.76522	292.7654079	292.7653607	292.7655016	292.7654662
h12	285.9643949	285.9647707	285.9646763	285.9649582	285.9648874	285.9650988
h13	278.6047694	278.6046436	278.6050193	278.604925	278.6052068	278.605136
h14	269.8987383	269.8991141	269.8990198	269.8993016	269.8992308	269.8994422
h15	259.1087172	259.1086543	259.1088422	259.108795	259.108936	259.1089006
h16	312.8638991	312.864087	312.8640398	312.8641807	312.8641453	312.864251
h17	307.7359925	307.7358982	307.73618	307.7361092	307.7363206	307.7362675
h18	301.1590336	301.1594094	301.159315	301.1595969	301.1595261	301.1597375
h19	293.1621289	293.1620345	293.1623164	293.1622456	293.162457	293.1624039
h20	283.1669294	283.1671173	283.1670701	283.167211	283.1671756	283.1672813
h21	335.9550725	335.955041	335.955135	335.9551114	335.9551818	335.9551642
h22	330.9562651	330.9564529	330.9564058	330.9565467	330.9565113	330.956617
h23	325.1347468	325.1346839	325.1348717	325.1348246	325.1349655	325.1349301
h24	318.4234368	318.4236247	318.4235775	318.4237184	318.423683	318.4237887
h25	310.397623	310.3975915	310.3976855	310.3976619	310.3977324	310.3977147

Well	t=79	t=80	t=81	t=82	t=83	t=84
h1	251.412919	251.4129058	251.4129454	251.4129354	251.4129652	251.4129577
h2	243.4187117	243.4187909	243.418771	243.4188305	243.4188155	243.4188601
h3	237.5087732	237.5087466	237.5088259	237.508806	237.5088654	237.5088505
h4	229.2192167	229.219296	229.2192761	229.2193355	229.2193206	229.2193652
h5	220.6471362	220.6471229	220.6471626	220.6471526	220.6471823	220.6471749
h6	272.2329113	272.2329906	272.2329707	272.2330302	272.2330152	272.2330598
h7	264.7534715	264.7534317	264.7535506	264.7535208	264.7536099	264.7535875
h8	257.3970581	257.3972167	257.3971769	257.3972957	257.3972659	257.3973551
h9	248.7212745	248.7212347	248.7213536	248.7213238	248.721413	248.7213906
h10	238.369275	238.3693543	238.3693343	238.3693938	238.3693789	238.3694235
h11	292.7655719	292.7655454	292.7656246	292.7656047	292.7656642	292.7656493
h12	285.9650457	285.9652042	285.9651644	285.9652833	285.9652535	285.9653426
h13	278.6053474	278.6052943	278.6054529	278.6054131	278.605532	278.6055021
h14	269.8993891	269.8995477	269.8995079	269.8996268	269.8995969	269.8996861
h15	259.1090063	259.1089797	259.109059	259.1090391	259.1090985	259.1090836
h16	312.8642245	312.8643037	312.8642838	312.8643433	312.8643284	312.8643729
h17	307.7364261	307.7363863	307.7365052	307.7364753	307.7365645	307.7365421
h18	301.1596844	301.1598429	301.1598031	301.159922	301.1598922	301.1599813
h19	293.1625624	293.1625226	293.1626415	293.1626117	293.1627008	293.1626784
h20	283.1672548	283.167334	283.1673141	283.1673736	283.1673587	283.1674033
h21	335.955217	335.9552037	335.9552434	335.9552334	335.9552631	335.9552557
h22	330.9565904	330.9566697	330.9566498	330.9567093	330.9566943	330.9567389
h23	325.1350358	325.1350093	325.1350885	325.1350686	325.1351281	325.1351131
h24	318.4237622	318.4238414	318.4238215	318.423881	318.423866	318.4239106
h25	310.3977675	310.3977542	310.3977939	310.3977839	310.3978136	310.3978062

Well	t=85	t=86	t=87	t=88	t=89	t=90
h1	251.41298	251.4129744	251.4129911	251.4129869	251.4129995	251.4129963
h2	243.4188489	243.4188824	243.418874	243.4188991	243.4188928	243.4189116
h3	237.5088951	237.5088839	237.5089173	237.5089089	237.508934	237.5089277
h4	229.219354	229.2193874	229.219379	229.2194041	229.2193978	229.2194166
h5	220.6471972	220.6471916	220.6472083	220.6472041	220.6472166	220.6472135
h6	272.2330486	272.2330821	272.2330737	272.2330987	272.2330924	272.2331113
h7	264.7536544	264.7536376	264.7536878	264.7536752	264.7537128	264.7537034
h8	257.3973327	257.3973996	257.3973828	257.3974329	257.3974203	257.3974579
h9	248.7214574	248.7214406	248.7214908	248.7214782	248.7215158	248.7215064
h10	238.3694123	238.3694457	238.3694373	238.3694624	238.3694561	238.3694749
h11	292.7656939	292.7656827	292.7657161	292.7657077	292.7657328	292.7657265
h12	285.9653202	285.9653871	285.9653703	285.9654205	285.9654079	285.9654455
h13	278.6055913	278.6055689	278.6056358	278.605619	278.6056691	278.6056565
h14	269.8996637	269.8997306	269.8997138	269.8997639	269.8997513	269.899789
h15	259.1091282	259.109117	259.1091504	259.109142	259.1091671	259.1091608
h16	312.8643618	312.8643952	312.8643868	312.8644119	312.8644056	312.8644244
h17	307.736609	307.7365922	307.7366423	307.7366297	307.7366673	307.7366579
h18	301.1599589	301.1600258	301.160009	301.1600592	301.1600466	301.1600842
h19	293.1627453	293.1627285	293.1627787	293.1627661	293.1628037	293.1627943
h20	283.1673921	283.1674255	283.1674171	283.1674422	283.1674359	283.1674547
h21	335.955278	335.9552724	335.9552891	335.9552849	335.9552974	335.9552943
h22	330.9567277	330.9567612	330.9567528	330.9567778	330.9567715	330.9567904
h23	325.1351577	325.1351465	325.13518	325.1351716	325.1351966	325.1351904
h24	318.4238994	318.4239329	318.4239245	318.4239496	318.4239433	318.4239621
h25	310.3978285	310.3978229	310.3978396	310.3978354	310.3978479	310.3978448

Well	t=91	t=92	t=93	t=94	t=95	t=96
h1	251.4130057	251.4130033	251.4130104	251.4130086	251.4130139	251.4130126
h2	243.4189068	243.418921	243.4189174	243.418928	243.4189253	243.4189333
h3	237.5089465	237.5089418	237.5089559	237.5089524	237.508963	237.5089603
h4	229.2194119	229.219426	229.2194225	229.219433	229.2194304	229.2194383
h5	220.6472229	220.6472205	220.6472276	220.6472258	220.6472311	220.6472298
h6	272.2331065	272.2331206	272.2331171	272.2331277	272.233125	272.233133
h7	264.7537316	264.7537245	264.7537456	264.7537403	264.7537562	264.7537522
h8	257.3974485	257.3974767	257.3974696	257.3974908	257.3974855	257.3975014
h9	248.7215346	248.7215275	248.7215487	248.7215434	248.7215592	248.7215553
h10	238.3694702	238.3694843	238.3694807	238.3694913	238.3694887	238.3694966
h11	292.7657453	292.7657406	292.7657547	292.7657511	292.7657617	292.7657591
h12	285.9654361	285.9654643	285.9654572	285.9654784	285.965473	285.9654889
h13	278.6056942	278.6056847	278.6057129	278.6057058	278.605727	278.6057217
h14	269.8997795	269.8998077	269.8998006	269.8998218	269.8998165	269.8998324
h15	259.1091796	259.1091749	259.109189	259.1091855	259.1091961	259.1091934
h16	312.8644197	312.8644338	312.8644302	312.8644408	312.8644382	312.8644461
h17	307.7366861	307.736679	307.7367002	307.7366949	307.7367108	307.7367068
h18	301.1600748	301.160103	301.1600959	301.1601171	301.1601117	301.1601276
h19	293.1628225	293.1628154	293.1628366	293.1628312	293.1628471	293.1628431
h20	283.16745	283.1674641	283.1674605	283.1674711	283.1674685	283.1674764
h21	335.9553037	335.9553013	335.9553084	335.9553066	335.9553119	335.9553106
h22	330.9567856	330.9567997	330.9567962	330.9568068	330.9568041	330.9568121
h23	325.1352092	325.1352044	325.1352185	325.135215	325.1352256	325.1352229
h24	318.4239574	318.4239715	318.4239679	318.4239785	318.4239758	318.4239838
h25	310.3978542	310.3978518	310.3978589	310.3978571	310.3978624	310.3978611

Well	t=97	t=98	t=99	t=100	t=101	t=102
h1	251.4130166	251.4130156	251.4130185	251.4130178	251.41302	251.4130195
h2	243.4189313	243.4189372	243.4189357	243.4189402	243.4189391	243.4189424
h3	237.5089682	237.5089662	237.5089722	237.5089707	237.5089752	237.508974
h4	229.2194363	229.2194423	229.2194408	229.2194452	229.2194441	229.2194475
h5	220.6472337	220.6472327	220.6472357	220.647235	220.6472372	220.6472366
h6	272.233131	272.2331369	272.2331354	272.2331399	272.2331388	272.2331421
h7	264.7537641	264.7537611	264.7537701	264.7537678	264.7537745	264.7537728
h8	257.3974974	257.3975093	257.3975063	257.3975152	257.397513	257.3975197
h9	248.7215672	248.7215642	248.7215731	248.7215709	248.7215775	248.7215759
h10	238.3694946	238.3695006	238.3694991	238.3695035	238.3695024	238.3695058
h11	292.765767	292.765765	292.765771	292.7657695	292.7657739	292.7657728
h12	285.9654849	285.9654968	285.9654938	285.9655028	285.9655005	285.9655072
h13	278.6057376	278.6057336	278.6057455	278.6057425	278.6057514	278.6057492
h14	269.8998284	269.8998403	269.8998373	269.8998462	269.899844	269.8998507
h15	259.1092013	259.1091993	259.1092053	259.1092038	259.1092083	259.1092071
h16	312.8644441	312.86445	312.8644486	312.864453	312.8644519	312.8644552
h17	307.7367187	307.7367157	307.7367246	307.7367224	307.7367291	307.7367274
h18	301.1601236	301.1601355	301.1601325	301.1601415	301.1601392	301.1601459
h19	293.162855	293.162852	293.162861	293.1628587	293.1628654	293.1628637
h20	283.1674744	283.1674804	283.1674789	283.1674833	283.1674822	283.1674855
h21	335.9553145	335.9553135	335.9553165	335.9553158	335.955318	335.9553174
h22	330.9568101	330.956816	330.9568145	330.956819	330.9568179	330.9568212
h23	325.1352309	325.1352289	325.1352348	325.1352333	325.1352378	325.1352367
h24	318.4239818	318.4239877	318.4239862	318.4239907	318.4239896	318.4239929
h25	310.397865	310.397864	310.397867	310.3978663	310.3978685	310.3978679

Well	t=103	t=104	t=105	t=106	t=107	t=108
h1	251.4130211	251.4130207	251.413022	251.4130217	251.4130226	251.4130224
h2	243.4189416	243.4189441	243.4189435	243.4189453	243.4189449	243.4189463
h3	237.5089774	237.5089766	237.5089791	237.5089784	237.5089803	237.5089798
h4	229.2194466	229.2194491	229.2194485	229.2194504	229.2194499	229.2194513
h5	220.6472383	220.6472379	220.6472391	220.6472388	220.6472398	220.6472395
h6	272.2331413	272.2331438	272.2331432	272.233145	272.2331446	272.233146
h7	264.7537779	264.7537766	264.7537804	264.7537794	264.7537822	264.7537815
h8	257.397518	257.397523	257.3975217	257.3975255	257.3975246	257.3975274
h9	248.7215809	248.7215796	248.7215834	248.7215824	248.7215853	248.7215846
h10	238.3695049	238.3695074	238.3695068	238.3695087	238.3695082	238.3695096
h11	292.7657761	292.7657753	292.7657778	292.7657772	292.7657791	292.7657786
h12	285.9655055	285.9655106	285.9655093	285.9655131	285.9655121	285.965515
h13	278.6057559	278.6057542	278.6057592	278.6057579	278.6057617	278.6057608
h14	269.899849	269.899854	269.8998527	269.8998565	269.8998556	269.8998584
h15	259.1092105	259.1092097	259.1092122	259.1092115	259.1092134	259.1092129
h16	312.8644544	312.8644569	312.8644563	312.8644582	312.8644577	312.8644591
h17	307.7367324	307.7367311	307.7367349	307.736734	307.7367368	307.7367361
h18	301.1601442	301.1601493	301.160148	301.1601518	301.1601508	301.1601536
h19	293.1628688	293.1628675	293.1628713	293.1628703	293.1628732	293.1628724
h20	283.1674847	283.1674872	283.1674866	283.1674885	283.167488	283.1674894
h21	335.9553191	335.9553187	335.95532	335.9553196	335.9553206	335.9553203
h22	330.9568204	330.9568229	330.9568223	330.9568241	330.9568237	330.9568251
h23	325.13524	325.1352392	325.1352417	325.1352411	325.1352429	325.1352425
h24	318.4239921	318.4239946	318.423994	318.4239959	318.4239954	318.4239968
h25	310.3978696	310.3978692	310.3978705	310.3978701	310.3978711	310.3978708

Well	t=109	t=110	t=111	t=112	t=113	t=114
h1	251.4130231	251.4130229	251.4130234	251.4130233	251.4130237	251.4130236
h2	243.4189459	243.418947	243.4189467	243.4189475	243.4189473	243.4189479
h3	237.5089813	237.5089809	237.508982	237.5089817	237.5089825	237.5089823
h4	229.219451	229.219452	229.2194518	229.2194526	229.2194524	229.219453
h5	220.6472402	220.6472401	220.6472406	220.6472405	220.6472409	220.6472408
h6	272.2331456	272.2331467	272.2331464	272.2331472	272.233147	272.2331476
h7	264.7537837	264.7537831	264.7537847	264.7537843	264.7537855	264.7537852
h8	257.3975267	257.3975288	257.3975283	257.3975299	257.3975295	257.3975306
h9	248.7215867	248.7215861	248.7215877	248.7215873	248.7215885	248.7215882
h10	238.3695093	238.3695103	238.3695101	238.3695108	238.3695106	238.3695112
h11	292.76578	292.7657797	292.7657807	292.7657804	292.7657812	292.765781
h12	285.9655142	285.9655164	285.9655158	285.9655174	285.965517	285.9655182
h13	278.6057636	278.6057629	278.605765	278.6057645	278.6057661	278.6057657
h14	269.8998577	269.8998598	269.8998593	269.8998609	269.8998605	269.8998616
h15	259.1092144	259.109214	259.1092151	259.1092148	259.1092156	259.1092154
h16	312.8644588	312.8644598	312.8644595	312.8644603	312.8644601	312.8644607
h17	307.7367382	307.7367377	307.7367393	307.7367389	307.73674	307.7367397
h18	301.1601529	301.1601551	301.1601545	301.1601561	301.1601557	301.1601569
h19	293.1628746	293.162874	293.1628756	293.1628752	293.1628764	293.1628761
h20	283.1674891	283.1674901	283.1674899	283.1674906	283.1674904	283.167491
h21	335.955321	335.9553209	335.9553214	335.9553213	335.9553217	335.9553216
h22	330.9568247	330.9568258	330.9568255	330.9568263	330.9568261	330.9568267
h23	325.1352439	325.1352435	325.1352446	325.1352443	325.1352451	325.1352449
h24	318.4239964	318.4239975	318.4239972	318.423998	318.4239978	318.4239984
h25	310.3978716	310.3978714	310.3978719	310.3978718	310.3978722	310.3978721

Well	t=115	t=116	t=117	t=118	t=119	t=120
h1	251.4130239	251.4130238	251.413024	251.413024	251.4130241	251.4130241
h2	243.4189478	243.4189482	243.4189481	243.4189484	243.4189484	243.4189486
h3	237.5089829	237.5089827	237.5089832	237.5089831	237.5089834	237.5089833
h4	229.2194528	229.2194533	229.2194532	229.2194535	229.2194534	229.2194537
h5	220.6472411	220.647241	220.6472412	220.6472411	220.6472413	220.6472413
h6	272.2331475	272.2331479	272.2331478	272.2331481	272.233148	272.2331483
h7	264.7537861	264.7537859	264.7537865	264.7537864	264.7537869	264.7537867
h8	257.3975303	257.3975312	257.397531	257.3975317	257.3975315	257.397532
h9	248.7215891	248.7215889	248.7215896	248.7215894	248.7215899	248.7215898
h10	238.3695111	238.3695115	238.3695114	238.3695118	238.3695117	238.3695119
h11	292.7657816	292.7657815	292.7657819	292.7657818	292.7657822	292.7657821
h12	285.9655179	285.9655188	285.9655186	285.9655193	285.9655191	285.9655196
h13	278.6057669	278.6057666	278.6057674	278.6057672	278.6057679	278.6057677
h14	269.8998613	269.8998622	269.899862	269.8998627	269.8998625	269.899863
h15	259.109216	259.1092158	259.1092163	259.1092162	259.1092165	259.1092164
h16	312.8644606	312.864461	312.8644609	312.8644613	312.8644612	312.8644614
h17	307.7367406	307.7367404	307.7367411	307.7367409	307.7367414	307.7367413
h18	301.1601566	301.1601575	301.1601573	301.1601579	301.1601578	301.1601583
h19	293.162877	293.1628768	293.1628775	293.1628773	293.1628778	293.1628777
h20	283.1674909	283.1674913	283.1674912	283.1674916	283.1674915	283.1674917
h21	335.9553219	335.9553218	335.955322	335.955322	335.9553221	335.9553221
h22	330.9568266	330.956827	330.9568269	330.9568272	330.9568271	330.9568274
h23	325.1352455	325.1352454	325.1352458	325.1352457	325.135246	325.1352459
h24	318.4239983	318.4239987	318.4239986	318.4239989	318.4239989	318.4239991
h25	310.3978724	310.3978723	310.3978725	310.3978725	310.3978726	310.3978726